

# PARTIAL-BURNSIDE GROUPS

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## CHAPTER I

### INTRODUCTION

#### I.1 History

The development of group theory has been greatly influenced by some questions about finitely generated periodic groups (a group is called periodic if every element has finite order) that Burnside asked in 1902 [Bu02]. Burnside asked whether every finitely generated periodic group is finite. He also asked the related question whether every finitely generated group satisfying the law  $x^n = 1$  (we say that such a group has exponent  $n$ ) is finite. Groups satisfying the law  $x^n = 1$  have come to be called Burnside groups of exponent  $n$ , and the variety of groups defined by this law is called the Burnside variety of exponent  $n$ . Free groups in this variety are called free Burnside groups, and the  $m$ -generator free Burnside group is denoted  $B(m, n)$ . Of course, the free Burnside groups of exponent 2 are abelian and are therefore well understood. In [Bu02], Burnside proved that  $B(m, 3)$  is finite and gave an upper bound for its order. Burnside [Bu05] showed that every finitely generated complex matrix group of finite exponent is finite, and Schur [S] went on to show that every finitely generated periodic group of complex matrices is finite.

Over the next fifty years, there was considerable progress for groups of small exponent. Levi and van der Waerden [LW] found that  $B(m, 3)$  has order  $3^{m + \binom{m}{2} + \binom{m}{3}}$ . In 1940, Sanov showed that  $B(m, 4)$  is finite. Marshall Hall [H] resolved the question in exponent 6, showing that  $B(m, 6)$  has order  $2^a 3^{b + \binom{b}{2} + \binom{b}{3}}$ , where  $a = 1 + (m-1)3^{m + \binom{m}{2} + \binom{m}{3}}$  and  $b = 1 + (m-1)2^m$ . It is worth pointing out that, with only a few exceptions, the order of  $B(m, 4)$  is unknown.

Finally, in 1964, Golod [Go] provided an example of a finitely generated periodic group that is infinite. However, the example of Golod contained elements of arbitrarily large order. In 1968, Novikov and Adian [NA] showed that  $B(m, n)$  is infinite if  $n \geq 4381$  is odd and  $m \geq 2$ . Adian [A75] later improved the result to hold for odd  $n \geq 665$ . In addition to showing infiniteness of free Burnside groups of large odd exponent, [NA] also illuminates much of the structure of these free Burnside groups. For instance, they show that every finite or abelian subgroup is cyclic. Additionally, they give an algorithm to solve the word problem in free Burnside groups of large odd exponent. In [O91], Olshanskii revisited the problem from a geometrical rather than combinatorial point of view; he gave a much simpler method for understanding free Burnside groups of large odd exponent (though his methods need a much larger exponent - greater than  $10^{80}$  suffices).

Dealing with free Burnside groups of even exponent proved to be much more difficult. In the 1990s, S. V. Ivanov and I. G. Lysenok proved that some free Burnside groups of even exponent are infinite. In particular, Ivanov [I94] showed that for  $m \geq 2$  and  $n \geq 2^{48}$  with  $n$  divisible by  $2^9$ , the group  $B(m, n)$  is infinite. Lysenok [L], showed that for  $m \geq 2$  and  $n = 16k \geq 8000$ ,  $B(m, n)$  is infinite.

In addition to the natural role that free Burnside groups have played in the study of finiteness of groups

of finite exponent, the Burnside variety has played an interesting role in the study of the finite basis problem - the question whether every variety of groups can be given by a finite set of laws. This problem was posed by B. H. Neumann [N.B] in 1937, and by 1967 products involving Burnside varieties had been identified as potential examples of varieties requiring infinitely many laws (see problem 11 of [N.H]). The finite basis problem was resolved ([A70], [O70], [V]) without any appeal to Burnside varieties (although Adian's proof used the methods of Novikov and Adian that had been developed to study the closely related free Burnside groups). However, Burnside varieties would soon take a more central role. Bryant [Br] and Kleiman [K73] independently proved that the product variety  $\mathfrak{B}_4\mathfrak{B}_2$  cannot be defined by finitely many laws. Kleiman [K74] later proved a stronger result: for any finitely based variety  $\mathfrak{V}$ , the product variety  $\mathfrak{B}_n\mathfrak{V}$  cannot be defined by a finite set of laws if  $n$  is divisible by the square of a prime that divides the exponent of  $\mathfrak{V}$ . Using his diagrammatic methods for free Burnside groups, Olshanskii [O91] showed that if  $p$  is a large prime, then  $\mathfrak{B}_p\mathfrak{A}$  is not finitely based, where  $\mathfrak{A}$  is the variety of abelian groups.

## I.2 Beyond Free Burnside Groups

The method of Novikov-Adian and the later diagrammatic method of Olshanskii have been applied to yield many results about periodic groups of large odd exponent, and to a lesser extent, some results about groups where every torsion element satisfies a law  $x^n = 1$  for some fixed large, odd  $n$ . However, it seems that there has been no systematic treatment of such groups. The intent of this work is to fill that gap. This work relies heavily on [O91].

Our primary object of study will be the following generalization of free Burnside groups:

**Definition I.1.** A group  $G$  is called a partial-Burnside group (of partial exponent  $n$ ) if

- i)  $G$  admits a presentation in terms of generators and relations in which every relator is an  $n$ -th power, and
- ii) every finite order element of  $G$  has order dividing  $n$ .

Examples of partial-Burnside groups are numerous:

*Example I.2.* The free group  $F(S)$  and free Burnside group  $B(S, n)$  with basis  $S$  are partial-Burnside groups.

*Example I.3.* If  $N \triangleleft F = F(S)$  is such that  $F/N$  is torsion-free, then  $F/N^n$  is partial-Burnside for large odd  $n$ , where  $N^n = \{X^n : X \in N\}$ . This is useful for some verbal subgroups  $N$ ; in particular, these groups will help us show that certain varieties  $\mathfrak{B}_n\mathfrak{V}$  are infinitely based.

*Example I.4.* Let  $F = F(S)$  denote the free group on  $S$ , and let  ${}^nF$  be the subgroup generated by the  $n$ th powers of words with the property that the exponents of some letter do not sum to a multiple of  $n$ . For large odd  $n$ , the group  $F/{}^nF$  is partial-Burnside.

Given the variation possible within the class of partial-Burnside groups, it is striking how frequently free and free Burnside groups occur as subgroups. Atabekian [A86], [A87] showed that every noncyclic subgroup of a free Burnside group of large odd exponent  $n$  contains a 2-generated free Burnside group of exponent  $n$ . Ivanov (Theorem 39.1 of [O91]) gave a very nice proof and also proved a stronger version [I03]. Our theorem I.6 is similar to Atabekyan's result, establishing when partial-Burnside groups contain

noncyclic free Burnside subgroups. The proof of the next theorem involves applying Ivanov's technique from [I03] to partial-Burnside groups.

**Theorem I.5.** *Let  $H$  be a noncyclic subgroup of a partial-Burnside group  $G$  of large, odd partial-exponent  $n$ . Then  $H$  contains a noncyclic partial-Burnside group of partial-exponent  $n$ .*

Immediate corollaries show that free and free Burnside subgroups of partial-Burnside groups are common.

**Corollary I.6.** *Every noncyclic torsion subgroup of a partial-Burnside group of sufficiently large odd partial exponent  $n$  contains a noncyclic free Burnside group of exponent  $n$ .*

**Corollary I.7.** *Every noncyclic torsion-free subgroup of a partial-Burnside group of sufficiently large odd partial exponent contains a noncyclic free subgroup.*

Furthermore, by viewing partial-Burnside groups as a limit of hyperbolic groups, we are able to establish another result about free subgroups of partial-Burnside groups.

**Theorem I.8.** *Every noncyclic finitely presented subgroup of a partial-Burnside group of sufficiently large odd partial exponent contains a noncyclic free subgroup.*

This theorem has an immediate corollary for free Burnside groups. The corollary is already known (see Remark 3.19 of [OOS], e.g.), but it is worth including as an application of the theory of partial-Burnside groups.

**Corollary I.9.** *Every finitely presented subgroup of a free Burnside group of sufficiently large odd exponent is cyclic.*

We also investigate some algorithmic questions in partial-Burnside groups. We compute filling length (introduced by Gromov in [Gr]) and isodiametric functions (introduced by Gersten in [Ge91]) associated to natural presentations of partial-Burnside groups. Finitely presented groups are known to have recursive isodiametric and filling length functions if and only if the word problem is solvable ([Ge], [GeR]). Although partial-Burnside groups are rarely finitely presentable, given any pair of words, we can easily find a finitely presented group which will detect whether they are equal (or conjugate) in a given partial-Burnside group. As a result:

**Proposition I.10.** *In a partial-Burnside group  $G$  of large, odd partial-exponent, the following are equivalent:*

- (1) *The set of words representing finite order elements of  $G$  is recursive.*
- (2) *The word problem is solvable in  $G$ .*
- (3) *The conjugacy problem is solvable in  $G$ .*

Not only can we study solvability of the word problem; we can also consider its complexity. The filling length function is an example of a space function ([O12]); its size is related to the space complexity of the word problem. Our results about filling length functions say that the word problem can be solved without having to store much information. More precisely:



**Proposition I.11.** *For free Burnside groups of large, odd exponent, the word problem has linear nondeterministic space complexity and polynomial deterministic space complexity.*

Finally, we turn our attention to products of varieties. Here we extend theorem 31.6 of [O91] and partially extend the results of [K74].

**Theorem I.12.** *If  $n$  is large and odd and  $\mathfrak{N}_c$  is the variety of groups of nilpotency class at most  $c$ , then  $\mathfrak{B}_n\mathfrak{N}_c$  is not finitely based.*

We also present joint work with Olshanskii, answering a question of Gupta and Krasinikov [GK]:

**Theorem I.13.** *If  $p$  is a large prime, then  $\mathfrak{B}_p\mathfrak{B}_p$  does not have a finite basis of laws.*

## CHAPTER II

### SOME PRELIMINARIES

This section introduces the necessary background results from [O91]. Whenever possible, the reader is referred to a specific location in [O91].

#### II.1 Words

**Definition II.1.** Let  $S$  be a set. We say that  $X$  is a word over  $S$  if every letter of  $X$  comes from the set  $S^{\pm 1} \cup \{1\}$ .

**Definition II.2.** Let  $X$  and  $Y$  be words over a set  $S$ . We say that  $X$  and  $Y$  are visually equal (and write  $X \equiv Y$ ) if, after deleting all 1's from  $X$  and  $Y$ , the resulting words consist of the same letters in the same order.

**Definition II.3.** A word  $Y$  is a cyclic shift of  $X$  if there are words  $U, V$  such that  $Y \equiv UV$  and  $X \equiv VU$ . A word over  $S$  is called reduced if it does not contain the letter 1 and it contains no subword  $xx^{-1}$  with  $x \in S \cup S^{-1}$ . A word is called cyclically reduced if all of its cyclic shifts are reduced words. A non-empty, cyclically reduced word  $X$  is called simple if there is no word  $Y$  such that  $X = Y^k$  in  $F(S)$  (the free group on  $S$ ) for some  $k > 1$ .

**Definition II.4.** A word  $X$  is called periodic with period  $A$  (also called  $A$ -periodic) if  $X$  is a subword of  $A^m$  for some  $m > 0$ . If the period  $A$  is a simple word, we say that  $X$  is periodic with simple period  $A$ .

Throughout this paper, we will consider only simple periods.

**Definition II.5.** Let  $X$  and  $Y$  be periodic words with simple period  $A$ . Then decompositions  $X \equiv X_1X_2$  and  $Y \equiv Y_1Y_2$  are called  $A$ -compatible if there exist an integer  $m > 0$  and a decomposition  $A^m \equiv U_1U_2$  such that  $U_1 \equiv V_1X_1 \equiv W_1Y_1$  and  $U_2 \equiv X_2V_2 \equiv Y_2W_2$  for some words  $V_1, V_2, W_1, W_2$ .

#### II.2 Maps

**Definition II.6.** Let  $X$  be a surface and  $P$  an  $k$ -gon ( $k \geq 2$ ) in the plane (if  $k = 2$ , we take  $P$  to be a disc with 2 points marked on its boundary). Let  $A_1A_2, \dots, A_kA_1$  be the sides of  $P$  and  $f$  be a continuous function  $P \rightarrow X$  such that

- 1)  $f|_{\text{int}(P)}$  is an embedding into  $X$ ,
- 2) the restriction to the interior of each side  $e = A_iA_{i+1}$  (with indices modulo  $k$ ) is an embedding into  $X$ , and
- 3) for distinct  $a, b \in P$  with  $f(a) = f(b)$ , we must have  $a, b \in \partial P$  and, moreover, if  $a$  is a vertex then so is  $b$ , and if  $a$  is not a vertex then the sides containing  $a$  and  $b$  have the same image in  $X$ .

Then,  $(\Pi, f)$  is called a cell with  $k$  sides, where  $\Pi = f(P)$ . Typically, we will simply say that  $\Pi$  is a cell, omitting explicit reference to the function  $f$  and the number  $k$ .

In the previous definition, by an edge, we mean the image of a side  $A_i A_{i+1}$ , together with a direction. More precisely, there is a continuous function  $g : [0, 1] \rightarrow A_i A_{i+1}$  which is a homeomorphism with  $\{g(0), g(1)\} = \{A_i, A_{i+1}\}$ . Then  $f \circ g$  parametrizes an edge  $e$  with initial endpoint  $e_- = f(g(0))$  and terminal endpoint  $e_+ = f(g(1))$ . The inverse edge of  $e$ , denoted  $e^{-1}$ , is parametrized by the function  $t \mapsto (f \circ g)(1 - t)$ . Clearly,  $(e^{-1})^{-1} = e$ . Two edges parametrized by  $f|_{A_i A_{i+1}} \circ g_0$  and  $f|_{A_i A_{i+1}} \circ g_1$  are equivalent if  $g_0^{-1} \circ g_1$  is an increasing function.

**Definition II.7.** Given a surface  $X$ , a cell decomposition is a finite set  $\{(\Pi_i, f_i)\}_{i=1}^m$  of cells such that

- 1)  $X = \bigcup_{i=1}^m \Pi_i$  and
- 2) the intersection of distinct cells  $\Pi_k, \Pi_l$  consists of common sides and common vertices (the intersection is empty if there are no common sides or vertices).

**Definition II.8.** Let  $X$  be a surface. We say that  $\Delta$  is a map on  $X$  if  $\Delta$  is a cell decomposition of  $X$ . Let  $p$  be some loop on  $X$  whose edges form the boundary of a subspace  $Y \subset X$  homeomorphic to a disc. Then the restriction of a cell decomposition  $\Delta$  to  $Y$  is a decomposition of  $Y$ , called a submap of the map  $\Delta$ . Maps on a disc, annulus, sphere, or torus will be called circular, annular, spherical, and toroidal, respectively.

By a path in  $\Delta$ , we mean a concatenation  $p = e_1 e_2 \cdots e_k$  of edges  $e_i$  with  $(e_i)_+ = (e_{i+1})_-$  for  $i = 1, \dots, k-1$ . The initial endpoint of  $p$  is  $p_- = (e_1)_-$  and its terminal endpoint is  $p_+ = (e_k)_+$ . We say that a path  $p$  is a closed path if  $p_- = p_+$ . For a path  $p = e_1 \cdots e_k$ , any path  $e_i e_{i+1} \cdots e_j$  with  $i \leq j$  is called a subpath of  $p$ . Additionally, if  $p$  is closed, we can regard  $p$  as a so-called cyclic path by identifying all of its cyclic shifts  $e_i e_{i+1} \cdots e_{i+k-1}$  where indices are taken modulo  $k$ . A subpath of a cyclic path is a subpath of any of its cyclic shifts.

**Definition II.9.** If  $\Delta$  is a map (or submap of another map) on a surface  $X$ , a contour of  $\Delta$  is a closed, cyclically reduced (i.e. containing no subpath  $ee^{-1}$ ) path in the boundary  $\partial\Delta$ . The contour of a circular map (submap)  $\Delta$  will be denoted by  $\partial\Delta$ . A subpath of a contour will be called a section (of a contour).

In a map, we partition the set of edges into 0-edges and  $S$ -edges. We require that the inverse of a 0-edge is a 0-edge. A cell will be called a 0-cell if either all the edges of its contour are 0-edges or its contour contains exactly two  $S$ -edges. For a path  $p$  in a map  $\Delta$ , we define the length  $|p|$  of  $p$  to be the number of  $S$ -edges in  $p$ .

**Definition II.10.** Two paths  $p$  and  $q$  in a map  $\Delta$  will be called combinatorially homotopic if we can pass from  $p$  to  $q$  using the following operations:

- 1)  $uv \rightarrow uee^{-1}v$  or  $uee^{-1}v \rightarrow uv$  where  $u, v$  are paths,  $e$  is an edge, and  $u_+ = e_- = v_-$ .
- 2)  $uv \rightarrow up_1 p_2 \cdots p_k v$  or  $up_1 p_2 \cdots p_k v \rightarrow uv$  where  $u, v$  are paths,  $p_1, \dots, p_k$  are edges with  $u_+ = v_- = (p_1)_- = (p_k)_+$ ,  $(p_i)_+ = (p_{i+1})_-$ , and  $p_1 p_2 \cdots p_k$  is the contour of a cell with  $k$  sides.

The path  $p$  is geodesic if  $|p| \leq |p'|$  for any path  $p'$  combinatorially homotopic to  $p$ .

**Definition II.11.** A graded map is a map  $\Delta$  together with a function

$$r : \{\text{cells of } \Delta\} \rightarrow \{0, 1, 2, \dots\}$$

such that  $r(\Pi) = 0 \Leftrightarrow \Pi$  is a 0-cell. We refer to  $r(\Pi)$  as the rank of  $\Pi$ .

Cells of positive rank will be called  $R$ -cells. We will denote the number of  $R$ -cells in a map  $\Delta$  by  $\Delta(2)$ .

The following three definitions, taken together, define contiguity submaps.

**Definition II.12.** Suppose that  $S$ -edges  $e, f$  belong to contours of  $R$ -cells  $\Pi_1, \Pi_2$  (or  $e$  belongs to the contour of an  $R$ -cell  $\Pi$  while  $f$  belongs to a section  $q$  of the contour of  $\Delta$ , or  $e$  and  $f$  belong to sections  $q_1, q_2$  of the contour of  $\Delta$ ) and that there is a sequence of 0-cells  $\pi_0, \pi_1, \dots, \pi_k$  with  $\partial\pi_i = e_i s_i e_{i+1}^{-1} p_i$ , with  $e_i$  and  $e_{i+1}$   $S$ -edges, and  $e_0 = e$  and  $e_{k+1} = f$ . Let  $s = s_0 s_1 \dots s_k$  and  $p = p_k p_{k-1} \dots p_0$ . Then  $p e s f^{-1}$  is the contour of a submap  $\Gamma$  of  $\Delta$ . This submap  $\Gamma$  is called a 0-bond between  $\Pi_1$  and  $\Pi_2$  (or between  $\Pi$  and  $q$  or between  $q_1$  and  $q_2$ ). We write  $e = \Gamma \wedge \Pi_1$  and  $f^{-1} = \Gamma \wedge \Pi_2$  (or  $e = \Gamma \wedge \Pi$  and  $f^{-1} = \Gamma \wedge q$  or  $e = \Gamma \wedge q_1$  and  $f^{-1} = \Gamma \wedge q_2$ ).

**Definition II.13.** Let  $k \geq 0$  and suppose we have defined  $j$ -bonds for all  $j \leq k$ . Suppose we have chosen a  $j$ -bond  $\Gamma_1$  and a  $k$ -bond  $\Gamma_2$  between  $R$ -cells  $\Pi_1$  and  $\Pi_2$  with  $j \leq k$  and either  $\Gamma_1 = \Gamma_2$  or  $\Gamma_1$  and  $\Gamma_2$  are disjoint. If  $\Gamma_1 = \Gamma_2$ , we call  $\Gamma = \Gamma_1$  the  $k$ -contiguity submap of  $\Pi_1$  to  $\Pi_2$  determined by the bond  $\Gamma_1 = \Gamma_2$ . If  $\Gamma_1, \Gamma_2$  are disjoint, we can write  $v_1 = \Gamma_1 \wedge \Pi_1$ ,  $v_2 = \Gamma_2 \wedge \Pi_1$ ,  $s_1 = \Gamma_1 \wedge \Pi_2$  and  $s_2 = \Gamma_2 \wedge \Pi_2$ , so that we can write  $\partial\Gamma_1$  and  $\partial\Gamma_2$  in the forms  $p_1 v_1 w_1 s_1$  and  $p_2 s_2 w_2 v_2$ , respectively. Let  $q_1$  (respectively  $q_2$ ) be a subpath in  $\partial\Pi_1$  (respectively in  $\partial\Pi_2$ ) having the form  $v_1 v v_2$  or  $v_2 v v_1$  (respectively,  $s_2 s s_1$  or  $s_1 s s_2$ ). If  $p_1 q_1 p_2 q_2$  (or  $w_2 q_1 w_1 q_2$ ) is the contour of a circular submap  $\Gamma$  containing neither  $\Pi_1$  nor  $\Pi_2$ , then we call  $\Gamma$  the  $k$ -contiguity submap of  $\Pi_1$  to  $\Pi_2$  defined by the bonds  $\Gamma_1$  and  $\Gamma_2$ . We say that the path  $q_1$  (or  $q_2$ ) is the contiguity arc of  $\Gamma$  to  $\Pi_1$  (or to  $\Pi_2$ ) and write  $q_1 = \Gamma \wedge \Pi_1$ ,  $q_2 = \Gamma \wedge \Pi_2$ . The ratio  $\frac{|q_1|}{|\partial\Pi_1|}$  is called the contiguity degree of  $\Pi_1$  to  $\Pi_2$  with respect to  $\Gamma$ , and is denoted  $(\Pi_1, \Gamma, \Pi_2)$  (if  $\Pi_1 = \Pi_2$ ,  $(\Pi_1, \Gamma, \Pi_2)$  is a pair of ratios). The paths  $p_1$  and  $p_2$  (or  $w_2$  and  $w_1$ ) are called the side arcs for  $\Gamma$ . We write  $\partial(\Pi_1, \Gamma, \Pi_2) = p_1 q_1 p_2 q_2$  (or  $\partial(\Pi_1, \Gamma, \Pi_2) = w_2 q_1 w_1 q_2$ ).

The definition is similar for  $k$ -contiguities of a cell to a section of  $\partial\Delta$ , a section of  $\partial\Delta$  to a cell, or one section of  $\partial\Delta$  to another.

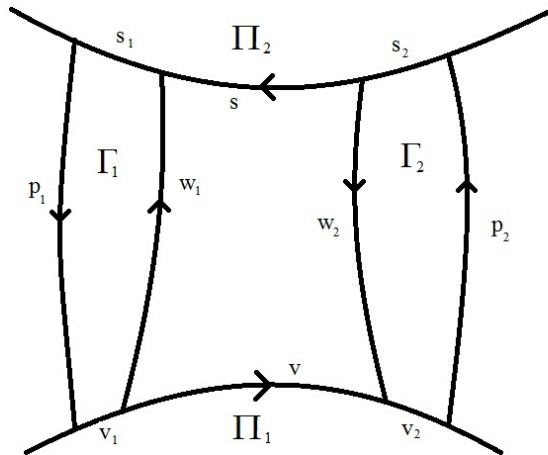


Figure II.1: A  $k$ -contiguity

Before making the next definition, we introduce a fixed constant  $\varepsilon \in (0, 1)$ .

**Definition II.14.** Let  $k > 0$  and suppose that we have defined  $j$ -contiguity submaps for all  $j < k$ . Let  $\pi, \Pi_1$ , and  $\Pi_2$  be cells (perhaps  $\Pi_1 = \Pi_2$ ) such that:

- 1)  $r(\pi) = k, r(\Pi_1) > k, r(\Pi_2) > k$ ,
- 2) there are disjoint submaps  $\Gamma_1$  and  $\Gamma_2$  of  $j_1$ -contiguity of  $\pi$  to  $\Pi_1$  and of  $j_2$ -contiguity of  $\pi$  to  $\Pi_2$ , respectively, with  $j_1 < k, j_2 < k$  such that  $\Pi_1$  is not contained in  $\Gamma_2$  and  $\Pi_2$  is not contained in  $\Gamma_1$ ,
- 3)  $(\pi, \Gamma_1, \Pi_1) \geq \varepsilon$  and  $(\pi, \Gamma_2, \Pi_2) \geq \varepsilon$ .

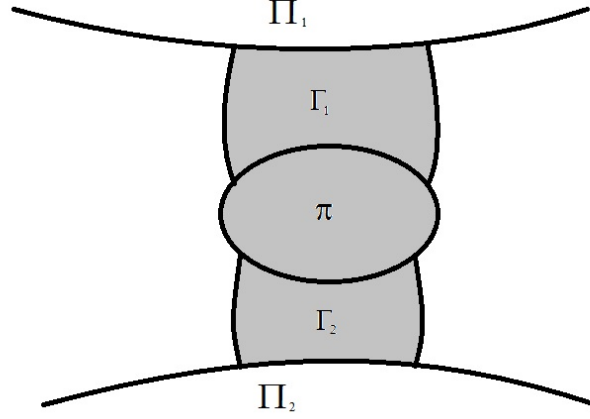


Figure II.2: A  $k$ -bond

Then there is a minimal submap  $\Gamma$  in  $\Delta$  containing  $\pi, \Gamma_1$ , and  $\Gamma_2$ ;  $\Gamma$  is called the  $k$ -bond between  $\Pi_1$  and  $\Pi_2$  defined by the contiguity submaps  $\Gamma_1$  and  $\Gamma_2$  with principal cell  $\pi$ . The contiguity arc of the bond  $\Gamma$  to  $\Pi_i$  is defined to be  $\Gamma_i \cap \Pi_i$ , and is denoted by  $\Gamma \wedge \Pi_i$ .

The definition is similar for  $k$ -bonds between a cell and a section of  $\partial\Delta$  or between two sections of  $\partial\Delta$ .

Typically, we will speak of a bond or a contiguity rather than a  $k$ -bond or a  $k$ -contiguity.

In this section, we will recall the important notion of  $A$ -maps from [O91]. We also state results that apply to what are called  $B$ -maps as well as to  $A$ -maps. We do not include the definition of  $B$ -maps. The reader may find the definition of  $B$ -maps in chapter 7 of [O91]. We also use the closely related notion of 'Condition R' without definition. This is a technical condition on the form of relations in a group presentation that allows the maps underlying a broad class of diagrams to be identified as  $B$ -maps. The reader may find the definition of condition R in chapter 8 of [O91].

In what follows, we make use of constants  $\alpha > \beta > \gamma > \delta > \varepsilon > \zeta > \iota_0 > \iota$  from the interval  $(0, 1)$ . These constants are chosen by the *least parameter principle* (LPP) to be a solution of a system of inequalities (some of the inequalities are given in [O91] and others are found in this paper). The process is carried out as follows: first, we choose  $\alpha$  to be small enough that all the inequalities involving only the parameter  $\alpha$  are true. Then, all of our inequalities involving only  $\alpha$  and  $\beta$  are true for  $\beta$  sufficiently small; so we choose  $\beta$  small

enough and continue. It is convenient to write  $\bar{\alpha} = \frac{1}{2} + \alpha$ ,  $\bar{\beta} = 1 - \beta$ ,  $\bar{\gamma} = 1 - \gamma$ ,  $h = \delta^{-1}$ ,  $n_0 = \iota^{-1}$ , and  $n = \iota^{-1}$ . Additionally, we choose  $\delta$ ,  $\iota_0$ , and  $\iota$  such that  $h$  is an integer and  $n$  and  $n_0$  are odd integers. At this stage our notation deviates from that of [O91]: we have interchanged the letters  $n_0$  and  $n$  in order to give our results a more consistent appearance.

**Definition II.15.** A (graded) map  $\Delta$  is called an  $A$ -map if:

(A1) The contour  $\partial\Pi$  of an arbitrary cell of rank  $j$  is cyclically reduced (i.e. contains no subpath of the form  $ee^{-1}$ ) and  $|\partial\Pi| \geq nj$ .

(A2) Any subpath of length  $\leq \max(j, 2)$  of the contour of an arbitrary cell of rank  $j$  in  $\Delta(2)$  is geodesic in  $\Delta$ .

(A3) If  $\pi, \Pi \in \Delta(2)$  and  $\Gamma$  is a contiguity submap of  $\pi$  to  $\Pi$  with  $(\pi, \Gamma, \Pi) \geq \varepsilon$ , then  $|\Gamma \wedge \Pi| < (1 + \gamma)r(\Pi)$ .

**Definition II.16.** A section  $q$  of a contour of a map  $\Delta$  is called a smooth section of rank  $k > 0$  (and we can write  $r(q) = k$ ) if:

(S1) every subpath of length  $\leq \max(k, 2)$  of  $q$  is geodesic in  $\Delta$ ;

(S2) for each contiguity submap  $\Gamma$  of a cell  $\pi$  to  $q$  satisfying  $(\pi, \Gamma, q) \geq \varepsilon$ , we have  $|\Gamma \wedge q| < (1 + \gamma)k$ .

**Lemma II.17.** [Lemma 15.1, [O91]] 1) A submap of an  $A$ -map is an  $A$ -map.

2) If a subpath  $p$  of a smooth section  $q$  of rank  $k$  of an  $A$ -map is a subpath of the contour of a submap  $\Gamma$ , then  $p$  can be regarded as a smooth section of rank  $k$  in  $\partial\Gamma$ .

3) If a subpath  $q$  of the contour of a cell  $\Pi$  of rank  $k$  is a section of the contour of a submap  $\Gamma$  of an  $A$ -map  $\Delta$  (and does not occur in  $\Gamma$ ), then  $q$  is a smooth section of rank  $k$  in  $\partial\Gamma$ .

**Lemma II.18.** [Lemma 15.3, [O91]] Let  $\Delta$  be an  $A$ -map,  $\Gamma$  a contiguity submap of a cell  $\Pi_1$  to a cell  $\Pi_2$  or to a section  $q$  of a contour, or of a section  $q^1$  to a section  $q^2$  of a contour. We set  $p_1q_1p_2q_2 = \partial(\Pi_1, \Gamma, \Pi_2)$  (or  $\partial(\Pi_1, \Gamma, q)$  or  $\partial(q^1, \Gamma, q^2)$ ) and  $P = \max(|p_1|, |p_2|)$ . Then

$$P < 2\varepsilon^{-1}r(\Pi_1) < \zeta nr(\Pi_1).$$

Additionally, in the first case  $P < \zeta nr(\Pi_2)$  while in the second and third cases, provided that  $q$  ( $q^1$  or  $q^2$ ) is smooth, we have  $P < \zeta nr(q)$  ( $P < \zeta nr(q^1)$  or  $P < \zeta nr(q^2)$ ).

**Lemma II.19.** [Lemma 15.4, [O91]] If the  $\Gamma$ -degree (i.e. the contiguity degree across the contiguity submap  $\Gamma$ ) of a cell  $\Pi_1$  to a cell  $\Pi_2$  (or to a section  $q$  of a contour) in an  $A$ -map is equal to  $\Psi$  and  $p_1q_1p_2q_2 = \partial(\Pi_1, \Gamma, \Pi_2)$  (or  $\partial(\Pi_1, \Gamma, q)$ ), then

$$(\bar{\beta} - 2\zeta\Psi^{-1})|q_1| < |q_2|.$$

In particular,  $|q_2| > (\Psi - 2\beta)|\partial\Pi_1|$ , and if  $\Psi \geq \varepsilon$ , then  $|q_1| < (1 + 2\beta)|q_2|$ .

Moreover, if  $q_2 = \Gamma \wedge \Pi_2$  (or  $q_2 = \Gamma \wedge q$  and  $q$  is a smooth section), then

$$|q_1| > \bar{\beta}(1 + 2\zeta\Psi^{-1})^{-1}|q_2|.$$

In particular, if  $\Psi \geq \varepsilon$ , then  $|q_1| > (1 - 2\beta)|q_2|$ .

**Lemma II.20.** [Lemma 15.6, [O91]] Let  $\Gamma$  be a contiguity submap of a cell  $\pi$  to a cell  $\Pi$  (or to a section  $q$  of a contour) in an A-map with  $(\pi, \Gamma, \Pi) \geq \bar{\alpha}$  (or  $(\pi, \Gamma, q) \geq \bar{\alpha}$ ). Further, let  $p_1 q_1 p_2 q_2 = \partial(\pi, \Gamma, \Pi)$  (or  $\partial(\pi, \Gamma, q)$ ) and let  $q'$  be the complement of  $q_1$  in  $\partial\pi$ . Then

$$|p_1(q')^{-1}p_2| < (1 + \gamma)|p_1(q')^{-1}p_2| < (1 - \gamma)|q_2| < |q_2|.$$

**Lemma II.21.** [Lemmas 15.8 and 21.7, [O91]] In any A-map or B-map, the contiguity degree of any cell to another cell or to a smooth section of the contour is less than  $\bar{\alpha}$ .

**Theorem II.22.** [Theorem 16.1, [O91]] Let  $\Delta$  be an A-map with  $r(\Delta) > 0$  on a surface  $S$  whose boundary is decomposed into at most  $l$  sections and  $\chi(S) \geq m$ . Then  $\Delta$  an R-cell  $\pi$  and distinguished contiguity submaps  $\Gamma_1, \dots, \Gamma_j$  such that that sum of the contiguity degrees of  $\pi$  across  $\Gamma_1, \dots, \Gamma_j$  is greater than  $\bar{\gamma}$ .

**Corollary II.23.** [Corollary 16.1, [O91]] Let  $\Delta$  be a circular A-map of nonzero rank whose contour is decomposed into sections  $q^1, q^2, q^3, q^4, q^5, q^6$  (or fewer sections). Then there is an R-cell  $\pi$  and disjoint contiguity submaps  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$  of  $\pi$  to  $q^1, q^2, q^3, q^4, q^5, q^6$ , respectively, in  $\Delta$  (some may be absent) such that

$$\sum_{i=1}^6 (\pi, \Gamma_i, q^i) > \bar{\gamma}.$$

**Corollary II.24.** [Corollary 16.2, [O91]] Let  $\Delta$  be an annular A-map of nonzero rank with contours  $p$  and  $q$ . Then there exist an R-cell  $\pi$  and disjoint contiguity submaps  $\Gamma_p, \Gamma_q$  of  $\pi$  to  $p$  and  $q$  respectively (one may be absent) with

$$(\pi, \Gamma_p, p) + (\pi, \Gamma_q, q) > \bar{\gamma}.$$

In the previous two results, the cell  $\pi$  is called a  $\gamma$ -cell.

The next result is similar to theorems 16.2 and 22.2 from [O91], and is actually proved in the course of proving these theorems.

**Theorem II.25.** Let  $\Delta$  be an A-map or B-map. If  $\Delta$  contains an R-cell  $\pi$  which is contiguous across  $\Gamma$  to a section  $q$  of  $\partial\Delta$ , with  $(\pi, \Gamma, q) > \varepsilon$ , then  $\Delta$  contains an R-cell  $\pi^*$  with a long section  $p$  and a contiguity submap  $\Gamma^*$  of  $p$  to  $q$  with  $r(\Gamma^*) = 0$  and  $(p, \Gamma^*, q) > \varepsilon$ . In the case when  $\Delta$  is an A-map, we take  $p = \partial\pi^*$  and have  $(\pi^*, \Gamma^*, q) > \varepsilon$ .

**Theorem II.26.** [Theorem 17.1, [O91]] Let  $\Delta$  be a circular A-map with contour  $qt$  or an annular A-map with contours  $q$  and  $t$ . If  $q$  is a smooth section, then  $\bar{\beta}|q| \leq |t|$  (equality holds if and only if  $|q| = |t| = 0$ ).

**Corollary II.27.** [Corollary 17.1, [O91]] If a circular A-map  $\Delta$  contains an R-cell  $\Pi$ , then  $|\partial\Delta| > \bar{\beta}|\partial\Pi|$ .

**Lemma II.28.** [Lemma 17.1, [O91]] Let  $\Delta$  be an annular A-map with contours  $p$  and  $q$  such that any loop consisting of 0-edges on the annulus is contractible to a point. Then there is a path  $t$  connecting vertices  $o_1$  and  $o_2$  of the paths  $p$  and  $q$ , respectively, such that  $|t| < \gamma(|p| + |q|)$ .

**Lemma II.29.** [Lemma 17.3, [O91]] Let  $\Delta$  be a circular A-map with contour  $p_1 q_1 p_2 q_2$ , where  $q_1$  and  $q_2$  are smooth sections of ranks  $k$  and  $l$ ,  $k \leq l$ , such that  $|p_1|, |p_2| < \alpha k$ . Then any vertex of  $q_1$  can be joined in  $\Delta$  to any vertex of  $q_2$  by a path  $y$  with  $|y| < \gamma^{-1}k$ .

**Lemma II.30.** [Lemma 17.4, [O91]] Let  $\Delta$  be a circular  $A$ -map with contour  $p_1q_1p_2q_2$ , where  $q_1$  and  $q_2$  are smooth sections of ranks  $k$  and  $l$ ,  $k \leq l$ , and  $|p_1|, |p_2| < \zeta nk$ . Then the perimeter of every cell  $\Pi$  in  $\Delta$  is less than  $3\gamma^{-1}\zeta nk < nk$  and  $r(\Delta) < 3\gamma^{-1}\zeta k < k$ .

**Lemma II.31.** [Lemma 17.5, [O91]] Let  $\Delta$  be a circular  $A$ -map with contour  $p_1q_1p_2q_2$ , where  $q_1$  and  $q_2$  are smooth sections of ranks  $k$  and  $l$ ,  $k \leq l$ , and  $|q_j| - M \geq 0$  for  $j = 1, 2$ , and  $M = \gamma^{-1}(|p_1| + |p_2| + k)$ . Then  $\Delta$  has a circular submap  $\Delta'$  with contour  $p'_1q'_1p'_2q'_2$  where  $q'_1(q'_2)$  is a subpath of  $q_1(q_2)$ ,  $|p'_1|, |p'_2| < \alpha k$ , and  $|q'_j| > |q_j| - M$  for  $j = 1, 2$ .

**Lemma II.32.** [Lemma 21.1, [O91]] Let  $\Delta$  be an  $A$ -map,  $\Gamma$  a contiguity submap of  $\bar{q}_1$  to  $\bar{q}_2$  with  $\partial(\bar{q}_1, \Gamma, \bar{q}_2) = p_1q_1p_2q_2$ . If  $\bar{q}_1$  is smooth of rank  $k$ , then  $|p_1|, |p_2| < 2h\epsilon^{-1}k$ .

**Definition II.33.** Let  $\Delta$  be a  $B$ -map on a sphere with three holes. Let  $q_1, q_2, q_3$  be the cyclic sections of the contour of  $\Delta$ . If  $q_1, q_2, q_3$  are smooth of ranks  $k, l, j$ , respectively, and

$$|q_1| > 100\zeta^{-1}k, |q_2| > 100\zeta^{-1}l, |q_3| > 100\zeta^{-1}j,$$

then we call  $\Delta$  a  $G$ -map.

**Lemma II.34.** [Lemma 24.8, [O91]] In any  $G$ -map  $\Delta$  with contour consisting of cyclic sections  $q_1, q_2$ , and  $q_3$ , there is a regular contiguity submap  $\Gamma$  of  $q_s$  to  $q_t$  (where  $s, t \in \{1, 2, 3\}$ ) such that  $(q_s, \Gamma, q_t) > \frac{1}{100}$  and  $(q_s^0, \Gamma, q_t^0) > \frac{1}{100}$ .

### II.3 Diagrams and Presentations

**Definition II.35.** Let  $\langle S|R \rangle$  be a presentation for a group  $G$ , where  $R$  consists of words that are not freely equal to 1, and let  $X$  be a surface. Suppose further that  $R$  is decomposed  $R = \bigcup_{i=1}^{\infty} R_i$ , where some of the  $R_i$  may be empty, and no word of  $R_i$  can coincide with a conjugate of a word from  $R_j$  or its inverse if  $i \neq j$  (such a presentation is called graded). A graded diagram  $\Delta$  consists of a graded map  $\Delta$  on  $X$  (note that we give the diagram the same name as the underlying map) together with a labelling function

$$Lab : \{\text{edges of } \Delta\} \rightarrow S^{\pm 1} \cup \{1\}$$

such that

- 1)  $Lab(e) = 1 \Leftrightarrow e$  is a 0-edge,
- 2)  $Lab(e^{-1}) = Lab(e)^{-1}$ , and
- 3) the label of any cell of rank  $i \geq 1$  is a cyclic shift of a word of  $R_i$  (or its inverse) and the label of a 0-cell is freely equal to 1.

We say that words  $X$  and  $Y$  are equal in rank  $j$  and write  $X =^j Y$  if  $X$  and  $Y$  are equal in the group  $\langle S | \bigcup_{i=1}^j R_i \rangle$ . If  $X = Y$  in the free group  $F(S)$ , we say that  $X$  and  $Y$  are equal in rank zero and write  $X =^0 Y$ . We say that  $X$  and  $Y$  are conjugate in rank  $j$  if  $X$  is equal in rank  $j$  to a conjugate of  $Y$ .



Since we defined 0-cells to be cells whose contours have exactly 0 or 2  $S$ -edges, conditions 1 and 3 in the definition above mean that, if a 0-cell in a graded diagram has an  $S$ -edge, then its contour has exactly two  $S$ -edges, and their labels are freely inverse to one another.

We will extend the labelling function  $Lab$  above to the set of paths in  $\Delta$  by declaring  $Lab(e_1 e_2 \cdots e_k) = Lab(e_1) Lab(e_2) \cdots Lab(e_k)$  for a path  $e_1 \cdots e_k$  in  $\Delta$ .

**Definition II.36.** We say that the diagram  $\Delta'$  is a 0-refinement of the diagram  $\Delta$  if the transition from  $\Delta'$  to  $\Delta$  can be achieved as a composition of the transitions of the following types:

1) If  $o$  is a vertex degree 2 and is a boundary vertex of two distinct sides  $e_1$  and  $e_2$ , with  $Lab(e_2) \equiv 1$  (or  $Lab(e_1) \equiv 1$ ), we delete  $o$  and replace  $e_1$  and  $e_2$  by a single edge  $e = e_1 \cup e_2$  with  $Lab(e) \equiv Lab(e_1)$  (or  $Lab(e) \equiv Lab(e_2)$ ) while leaving the rest of the diagram unchanged.

2) If  $o$  is a vertex of a cell  $\Pi$  with  $k$  sides ( $k \geq 3$ ) with  $o$  having degree 1, and  $o$  is an endpoint of an edge  $e$  with  $Lab(e) \equiv 1$ , we can delete all of  $e$  except for its other endpoint, while leaving the rest of the diagram unchanged.

3) If  $\Pi_1$  is a cell with  $k$  sides,  $\Pi_2$  is a cell with  $l$  sides, with  $k + l > 2$  and  $\Pi_1, \Pi_2$  share a edge  $e$  but  $\Pi_1 \neq \Pi_2$ , with one of  $\Pi_1, \Pi_2$  a cell of rank 0, we can delete the edge  $e$  (except for endpoints), leaving the rest of the diagram unchanged.

If  $\Pi$  is a cell in a diagram  $\Delta$ , a cell  $\Pi'$  in a 0-refinement  $\Delta'$  of  $\Delta$  is called a copy of  $\Pi$  if  $S$ -edges of  $\Pi$  and  $\Pi'$  all occur as pairs in 0-cells of  $\Delta'$ .

**Definition II.37.** A (graded) diagram  $\Delta$  is called reduced if there is no 0-refinement  $\Delta'$  of  $\Delta$  containing copies  $\Pi'_1, \Pi'_2$  of a pair of cells  $\Pi_1, \Pi_2$  of rank  $j \geq 1$  and a path  $t$  without self-intersection joining vertices  $o_1, o_2$  of  $\Pi'_1, \Pi'_2$  in  $\Delta'$  such that  $\phi(t) = j^{-1} 1$  and the contour labels  $p_1, p_2$  of  $\Pi'_1, \Pi'_2$  (where  $(p_1)_- = o_1, (p_2)_- = o_2$ ) are inverse in the free group  $F(S)$ . The cells  $\Pi_1$  and  $\Pi_2$  described above form what is called a  $j$ -pair.

**Definition II.38.** Let  $\Delta$  be a diagram over a graded presentation. We define the type  $\tau(\Delta)$  of  $\Delta$  to be the tuple  $(\tau_1, \tau_2, \dots)$  where  $\tau_i$  is the number of cells of rank  $i$  in  $\Delta$ . We declare  $\tau(\Delta) < \tau(\Delta')$  if there is  $i$  such that  $\tau_i(\Delta) < \tau_i(\Delta')$  and  $\tau_j(\Delta) = \tau_j(\Delta')$  for all  $j > i$ .

*Remark II.39.* If  $\Delta$  is a diagram, and  $\Delta$  has the smallest type among all diagrams sharing its contour label(s), then  $\Delta$  is reduced. We say that such a diagram  $\Delta$  has minimal type (or simply "is minimal").

**Theorem II.40.** [Theorem 13.1, [O91]] Let  $W$  be a non-empty word over  $S$ . Then  $W = 1$  in the group  $G$  with graded presentation

$$\left\langle S \mid \bigcup_{i=1}^{\infty} R_i \right\rangle \quad (\text{II.1})$$

if and only if there is a reduced graded circular diagram over (II.1) whose contour label is visually equal to  $W$ .

**Theorem II.41.** [Theorem 13.2, [O91]] Let  $V$  and  $W$  be non-empty words over  $S$ . Then they are conjugate in the group  $G$  given by (II.1) if and only if there is a reduced graded annular diagram over (II.1) with contours  $p$  and  $q$  such that  $Lab(p) \equiv V$  and  $Lab(q) \equiv W^{-1}$ .

*Remark II.42.* Suppose that  $\Pi_1, \Pi_2$  form a  $j$ -pair in a diagram  $\Delta$ . Let  $\Pi'_1, \Pi'_2$  be copies of  $\Pi_1, \Pi_2$  as in definition II.37 and  $t$  be as in that definition as well. We 0-refine to obtain a diagram  $\Delta'$ , and write  $\partial\Pi'_1 = p_1 p'_1$ ,  $\partial\Pi'_2 = p_2 p'_2$ , and there is a circular subdiagram  $\Gamma$  with contour  $(p'_1)^{-1} t' (p'_2)^{-1} (t'')^{-1}$ , where  $Lab(t') \equiv Lab(t'') \equiv Lab(t) = {}^{j-1}1$ . Now, let  $\Gamma'$  be the subdiagram consisting of  $\Pi'_1, \Pi'_2$ , and  $\Gamma$ ; its contour has label

$$Lab(p_1)Lab(t')Lab(p_2)Lab(t'')^{-1} = {}^{j-1}1 \quad Lab(p_1)Lab(p_2) = {}^01.$$

By theorem II.40, there is a circular diagram  $\bar{\Delta}$  of rank  $j-1$  with contour label  $Lab(p_1)Lab(t')Lab(p_2)Lab(t'')^{-1}$ . Replacing  $\Gamma'$  by  $\bar{\Delta}$  in  $\Delta$ , we obtain a diagram with smaller type than  $\Delta$  and the same contour label(s).

**Definition II.43.** Let  $q^1$  and  $q^2$  be sections of a contour of a diagram  $\Delta$  of rank  $i$  such that  $Lab(q^1)$  and  $Lab(q^2)$  are periodic words with periods  $A$  and  $A^{-1}$ , where  $A$  is a simple word. We say that  $q^1$  and  $q^2$  are  $A$ -compatible if there are vertices  $o_1, o_2$  on  $q^1, q^2$  defining  $A$ -compatible decompositions of  $Lab(q^1)$  and  $Lab((q^2)^{-1})$  as in definition II.5, and there is a path  $t$  without self-intersection in  $\Delta$  such that  $t_1 = o_1$ ,  $t_+ = o_2$ , and  $Lab(t) = {}^i1$ , and, moreover,  $Lab(t) = {}^{j-1}1$  if  $A^k$  labels a cell of rank  $j$ , for some  $k$ .

The definition of the  $A$ -compatibility of a cell of rank  $j$  with contour label of the form  $A^{\pm n}$  with a section  $q^1$  of a contour of a diagram is defined by replacing  $q^2$  above with the contour  $\partial\Pi$  of a cell  $\Pi$ .

**Lemma II.44.** [Lemma 13.3, [O91]] Let  $\Delta$  be a reduced graded diagram on a surface  $X$  whose contour decomposes into sections  $p_1, p_2, \dots$  with  $Lab(p_1) \equiv A^l$  for some period  $A$  of rank  $j$  and let  $\Pi$  be a cell  $A$ -compatible with  $p_1$  in  $\Delta$  and  $|\partial\Pi| = n|A|$ . Then there is a reduced diagram  $\Delta'$  on  $X$  whose contour decomposes into sections  $p'_1, p'_2, \dots$  in such a way that  $\tau(\Delta') < \tau(\Delta)$ ,  $Lab(p'_i) \equiv Lab(p_i)$  for  $i \geq 2$ ,  $Lab(p'_1) \equiv A^{l+ns}$  for some integer  $s$ , and in  $\Delta'$ , as well as in any 0-refinement of it, there is no cell  $A$ -compatible with  $p'_1$ .

## II.4 A graded presentation for partial-Burnside groups

In order to work with partial-Burnside groups, it is useful to have a more concrete characterization. Toward this end, we need a few definitions.

**Definition II.45.** A set  $C$  of words in the alphabet  $S$  is called partial-Burnside (of partial-exponent  $n$ ) if

- (C1) if  $w = u$  in the free group  $F(S)$ , then  $w \in C \Leftrightarrow u \in C$ ,
- (C2) for  $m$  a nonzero integer,  $zw^mz^{-1} \in C \Rightarrow w \in C$ , and
- (C3) if  $w \in C$ , then  $w^n u \in C \Leftrightarrow u \in C$ .

**Definition II.46.** Given a partial-Burnside set (of partial-exponent  $n$ ) in the alphabet  $S$ ,  $B_C(S, n)$  is the group with presentation

$$\langle S | R^n = 1, R \in C \rangle. \quad (\text{II.2})$$

Likewise, define  $B_C^k(S, n) = \langle S | R^n = 1, R \in C, |R| \leq k \rangle$ .

A presentation of the form (II.2) is called a partial-Burnside presentation.

Partial-Burnside groups of large, odd partial-exponent  $n$  are precisely the groups  $B_C(S, n)$  with  $C$  a partial-Burnside set, as we will prove in proposition III.4. For now, we must content ourselves with a graded presentation of  $B_C(S, n)$ .

**Definition II.47.** Let  $C$  be a partial-Burnside set of words over  $S$ . Then we define the group  $G_C(\infty)$  as follows:

Let  $R_0 = \emptyset$  and  $G_C(0) = F(S)$ , the free group on  $S$ . Suppose that  $R_{i-1}$ , and  $G_C(i-1)$  have been defined and also that a set  $L_j$  (its elements are called periods of rank  $j$ ) has been defined for each  $j < i$ .

Then let  $L_i$  (the set of periods of rank  $i$ ) be a maximal set of words of  $C$  satisfying the following three conditions:

- (1) Each element of  $L_i$  has length  $i$ .
- (2) If  $A \in L_i$ , then  $A$  is not conjugate in rank  $i-1$  to a power of a word shorter than  $A$  or to a power of a period of rank less than  $i$ .
- (3) If  $A, B \in L_i$  with  $A \neq B$  (visually) then  $A$  is not conjugate in rank  $i-1$  to  $B$  or  $B^{-1}$ .

Set  $R_i = \{A^n : A \in L_i\} \cup R_{i-1}$  and  $G_C(i) = \langle S | R_i \rangle$ . Take  $R = \bigcup_{i=1}^{\infty} R_i$  and set  $G_C(\infty) = \langle S | R \rangle$ .

**Definition II.48.** A non-trivial (in  $F(S)$ ) word  $X$  is said to be simple in rank  $i \geq 0$  if it is not conjugate in rank  $i$  to any power  $Y^m$  of a period of rank  $k \leq i$  and not conjugate in rank  $i$  to a power of any word  $Z$  with  $|Z| < |X|$ .

**Lemma II.49.** [Lemma 18.1, [O91]] Every word is conjugate in rank  $i \geq 0$  to a power of some period of rank  $j \leq i$  or to a power of a word simple in rank  $i$ .

Lemmas II.50 through II.55 and theorems II.56 and II.57 below are analogs of results from chapter 6 of [O91]; the only difference is that the results in [O91] are stated for the group  $G_{F(S)}(\infty)$ . The statements of the auxilliary lemmas in [O91] and all the proofs are valid, where the reader needs only to keep in mind that all periods are chosen inside  $C$ . The essential features of the periods are that they are cyclically reduced and simple in smaller ranks. These properties follow from the conditions on  $C$  and the method for selecting periods.

**Lemma II.50.** [Corollary 18.2, [O91]] If  $XY =^j YX$ , then there is a word  $Z$  such that  $X =^j Z^k$  and  $Y =^j Z^l$  for some integers  $k$  and  $l$ .

**Lemma II.51.** [Lemma 18.3, [O91]] If  $X \neq^i 1$  and  $X$  has finite order in rank  $i$ , then it is conjugate in rank  $i$  to a power of some period of rank  $\leq i$ .

**Lemma II.52.** [Lemma 18.7, [O91]] Let  $Z_1 A^{m_1} Z_2 =^i A^{m_2}$ ,  $m = \min(m_1, m_2)$ , where  $A$  is simple in rank  $i$ . If  $|Z_1| + |Z_2| < (\gamma(m - \frac{5}{6}h - 1) - 1)|A|$ , then  $Z_1$  and  $Z_2$  are equal in rank  $i$  to powers of  $A$ .

The next result is slightly modified from the version given in [O91], but its proof is identical, provided the corresponding changes are made in the statements of lemmas 18.6 and 18.8.

**Lemma II.53.** [Lemma 18.9, [O91]] Let  $\Delta$  be a reduced circular diagram of rank  $i$  with contour  $p_1 q_1 p_2 q_2$  where  $\text{Lab}(q_1)$  and  $\text{Lab}(q_2)$  are periodic words with period  $A$  simple in rank  $i$ ,  $|p_1| < 2\alpha|A|$ , and  $|p_2| < (\frac{1}{2} + 2\alpha)|A|$ . Then  $|q_1|, |q_2| \leq h|A|$ .

**Lemma II.54.** [Lemma 19.1, [O91]] Let  $\Delta$  be a reduced circular diagram of rank  $i$  with contour  $p_1 q_1 p_2 q_2$ , where  $\text{Lab}(q_1), \text{Lab}(q_2)$  are periodic words with periods  $A, B$  simple in rank  $i$ , and suppose that  $|p_1|, |p_2| < \alpha|B|$ ,  $|q_1| > (1 + \frac{1}{2}\gamma)|A|$ , and  $|q_2| > \frac{1}{2}\epsilon n|B|$ . If  $\text{Lab}(q_1)$  begins with  $A$  and  $\text{Lab}(q_2^{-1})$  begins with  $B^{-1}$ , then  $A =^i \text{Lab}(p_1)^{-1} B^{\pm 1} \text{Lab}(p_1)$ .

**Lemma II.55.** [Lemma 19.2, [O91]] Let  $\Delta$  be a reduced circular diagram of rank  $i$  with contour  $p_1 q_1 p_2 q_2$ , where  $\text{Lab}(q_1), \text{Lab}(q_2)$  are periodic words with periods  $A, B$  simple in rank  $i$ , and  $|q_2| \geq \epsilon n|B|$  and  $|p_1|, |p_2| < \zeta n c$  with  $c = \min(|A|, |B|)$ . Then either  $|q_1| < (1 + \gamma)|A|$  or  $A$  is conjugate in rank  $i$  to  $B^{\pm 1}$ .

**Theorem II.56.** [Lemma 19.4, [O91]] Every reduced diagram  $\Delta$  over  $G_C(\infty)$  is an  $A$ -map.

**Theorem II.57.** [Lemma 19.5, [O91]] Let  $p$  be a section of the contour of a reduced diagram  $\Delta$  whose label is  $A$ -periodic, where  $A$  is simple in rank  $r(\Delta)$  or is a period of rank  $k \leq r(\Delta)$ , and in the latter case  $\Delta$  has no cells of rank  $k$   $A$ -compatible with  $p$ . (If  $p$  is a cyclic section, then we further require that  $\text{Lab}(p) \equiv A^m$  for some integer  $m$ ). Then  $p$  is a smooth section of rank  $|A|$  in the contour of  $\Delta$ .

**Theorem II.58.** [Lemma 26.5, [O91]] Every reduced diagram on a disc, annulus, or sphere with three holes over a graded presentation satisfying condition  $R$  is a  $B$ -map.

**Lemma II.59.** [Lemma 25.11, [O91]] If a group  $G$  is given by a graded presentation  $G(\infty)$  satisfying condition  $R$ , then every period of the presentation is nontrivial in  $G$ .

**Lemma II.60.** [Lemma 25.19, [O91]] Let  $\Delta$  be a reduced diagram of rank  $i$  on a sphere with three holes with contours  $q_1^0, q_2^0$ , and  $q_3^0$ . Let  $q_1^0$  and  $q_2^0$  have labels  $A^{m_1}$  and  $A^{-m_1}$  (or  $A^{-m_1+vn}$  if  $A$  is a period of rank  $\leq i$ ) respectively,  $m_1 > 100\zeta^{-1}$  and either  $A$  is simple in rank  $i$  or a period of rank  $\leq i$ . Suppose that  $q_3^0$  has label  $B^{m_2}$ , where  $B$  is either simple in rank  $i$  or is a period of rank  $\leq i$ . If the contours of  $\Delta$  are smooth, then either  $|m_2| \leq 100\zeta^{-1}$ , or, when  $A \equiv B$  is a period of rank  $\leq i$ , we have  $|m_2 + un| \leq 100\zeta^{-1}$  for some  $u$ .

**Lemma II.61.** [Lemma 25.20, [O91]] If the hypotheses of Lemma II.60 are satisfied and  $m_2 \neq 0$  (or  $m_2 \neq vn$  if  $B$  is a period of rank  $\leq i$ ), then  $|B^{m_2}| \geq \zeta|A^{m_1}|$ .

**Lemma II.62.** [Lemma 25.10, [O91]] Let  $\Delta$  be a reduced circular diagram of rank  $i$  with contour  $p_1 q_1 p_2 q_2$ , where  $\text{Lab}(q_1)$  and  $\text{Lab}(q_2)$  are periodic words with periods  $A$  and  $B$  simple in rank  $i$ ,  $|q_2| \geq \zeta^{-1}|B|$ ,  $\max(|p_1|, |p_2|) < 2h\epsilon^{-1}c$ , and  $c = \min(|A|, |B|)$ . Then either  $|q_1| < (1 + \gamma)|A|$  and  $|A| > |B|$  or  $A$  is conjugate in rank  $i$  to  $B^{\pm 1}$ , and if  $A \equiv B^{\pm 1}$ , then  $A \equiv B^{-1}$  and  $q_1$  and  $q_2$  are  $A$ -compatible.

**Lemma II.63.** [Lemma 25.8, [O91]] Let  $\Delta$  be a reduced circular diagram of rank  $i$  with contour  $p_1 q_1 p_2 q_2$ , where  $\text{Lab}(q_1)$  and  $\text{Lab}(q_2)$  are periodic words with period  $A$  simple in rank  $i$ . If  $\max(|p_1|, |p_2|) < 2h\epsilon^{-1}|A|$ , then  $\max(|q_1|, |q_2|) < \zeta^{-1}|A|$ .

The next result follows immediately from corollary 31.1 of [O91].

**Theorem II.64.** Let  $N$  be the kernel of the projection  $F(S)$  onto a partial-Burnside group  $B_C(S, n)$ . Let  $\mathfrak{R}$  be the set of periods in a graded presentation of  $B_C(S, n)$ . Then  $N/[F, N]$  is a free abelian group with basis  $\{R^n[F, N] : R \in \mathfrak{R}\}$ .

## CHAPTER III

### PARTIAL-BURNSIDE GROUPS AND $B_C(S, N)$

Our main goal in this section is to show that partial-Burnside groups of large odd partial-exponent  $n$  are precisely the groups  $B_C(S, n)$ , where  $C$  is a partial-Burnside set. To do this, we will first show that our graded presentation  $G_C(\infty)$  defines the group  $B_C(S, n)$ .

**Lemma III.1.** *Suppose  $X = ZY^kZ^{-1}$  in  $G_C(j)$ , where  $C$  is partial-Burnside and  $k \neq 0$ . If  $X \in C$  then  $Y \in C$ .*

*Proof.* If  $X = ZY^kZ^{-1}$  in  $G_C(j)$ , then  $X = ZY^kZ^{-1} \prod V_i R_i^{\pm n} V_i^{-1}$  in the free group, where  $R_i^n = 1$  is a relation in the presentation of  $G_C(j)$  (so  $R_i \in C$ ). If  $X \in C$ , it follows from condition C1 that

$$ZY^kZ^{-1} \prod V_i R_i^{\pm n} V_i^{-1} \in C.$$

Since  $R_i \in C$ , it follows from C1 and C2 that  $V_i R_i^{\pm 1} V_i^{-1} \in C$  since for  $\varepsilon = \pm 1$ ,

$$V_i^{-1} (V_i R_i^{\varepsilon} V_i^{-1})^{\varepsilon} V_i = R_i \in C.$$

Hence, by repeated applications of C3,

$$ZY^kZ^{-1} \prod V_i R_i^{\pm n} V_i^{-1} \in C \Rightarrow ZY^kZ^{-1} \in C.$$

Now,  $Y \in C$  by an immediate application of C2. □

**Proposition III.2.** *1) If  $X \neq 1$  has finite order in  $G_C(\infty)$  (in  $G_C(j)$ ), then  $X$  is equal in  $G_C(\infty)$  (in  $G_C(j)$ ) to a power of an element of  $C$  (an element of  $C$  with length  $\leq j$ ).*

*2) If  $X \in C$  and  $|X| = i$ , then  $X^n =^i 1$ .*

*3) If  $X$  is conjugate in  $G_C(\infty)$  (in  $G_C(j)$ ) to a power of an element of  $C$  (having length  $\leq j$ ), then  $X^n = 1$  in  $G_C(\infty)$  (in  $G_C(j)$ ).*

*Proof.* 1) Suppose  $X \neq 1$  has finite order in  $G_C(\infty)$ . Then  $X$  has finite order in  $G_C(j)$ , for some  $j$ . By lemma II.51,  $X =^j ZY^kZ^{-1}$  with  $Y$  a period, and therefore  $Y \in C$ . By C1 and C2,  $ZYZ^{-1} \in C$  since  $Z^{-1}(ZYZ^{-1})Z = Y \in C$ .

2) If  $i = 0$ , then  $X \equiv 1$ , so  $X^n =^0 1$ . Suppose now that  $i > 0$  and that  $T^n =^j 1$  for all  $T \in C$  with  $|T| = j < i$ . By definition II.48 and the fact that periods of rank  $j$  have length  $j$ , it follows that  $X$  is simple in rank  $i - 1$  or  $X =^{i-1} ZY^kZ^{-1}$  with  $|Y| < |X|$ . If  $X$  is simple in rank  $i - 1$ , then  $X =^{i-1} PB^{\pm 1}P^{-1}$  for some  $B \in L_i$ , and it follows that  $X^n =^{i-1} PB^{\pm n}P^{-1} =^i PP^{-1} =^0 1$ . On the other hand, suppose  $X =^{i-1} ZY^kZ^{-1}$  with  $|Y| < |X| = i$ . By lemma III.1,  $Y \in C$ . Hence, by the induction hypothesis,  $X^n =^{i-1} ZY^{nk}Z^{-1} =^{i-1} ZZ^{-1} =^0 1$ . In either case,  $X^n =^i 1$ .

3) If  $X = ZY^kZ^{-1}$  in  $G_C(\infty)$  (in  $G_C(S, n)$ ) with  $Y \in C$  (with  $|Y| \leq j$ ), then by part 2),

$$X^n = (ZY^kZ^{-1})^n = Z(Y^n)^kZ^{-1} = ZZ^{-1} = 1$$

holds in  $G_C(\infty)$  (in  $G_C(j)$ ). □

**Theorem III.3.**  $B_C(S, n) \cong G_C(\infty)$  and  $B_C^k(S, n) \cong G_C(k)$ .

*Proof.* In our presentation for  $B_C(S, n)$ , the relators were  $n$ -th powers of elements of  $C$ . By proposition III.2,  $X^n = 1$  in  $G_C(\infty)$  if  $X \in C$ . By Dyck's theorem, the identity function on  $S$  extends to a homomorphism  $g : B_C(S, n) \rightarrow G_C(\infty)$ .

Similarly, the identity function on  $S$  extends to a homomorphism  $h : G_C(\infty) \rightarrow B_C(S, n)$ . Since the compositions  $h \circ g$  and  $g \circ h$  are both the identity on the generating set  $S$  (it's a generating set for both  $B_C(S, n)$  and  $G_C(\infty)$ ), it follows that  $h \circ g$  and  $g \circ h$  are isomorphisms.

Similarly,  $B_C^k(S, n) \cong G_C(k)$ . □

In what follows, whenever we speak of equality (or conjugacy) in rank  $j$  in  $B_C(S, n)$ , we mean equality (or conjugacy) in  $G_C(j)$ . Similarly, when we speak of periods of  $B_C(S, n)$ , we mean periods in a graded presentation  $G_C(\infty)$ .

**Proposition III.4.** A group  $G$  is partial-Burnside of partial exponent  $n$  if and only if  $G \cong B_C(S, n)$  for some set  $S$  and some partial-Burnside set  $C$  in the alphabet  $S$ .

*Proof.* First, we will prove that  $B_C(S, n)$  is partial-Burnside. By proposition III.2, every finite order element has order dividing  $n$ . By definition,  $B_C(S, n)$  has a presentation in which every relator is an  $n$ th power. Thus,  $B_C(S, n)$  is a partial-Burnside group.

Suppose now that  $G$  is a partial-Burnside group. Then  $G$  has a presentation  $G \cong \langle S | \mathfrak{R} \rangle$  in which every relator is an  $n$ th power. Let  $C$  be the set of words in the alphabet  $S$  which represent finite order elements of  $G$ . We must check that  $C$  is a partial-Burnside set. Condition C1 is trivial. If  $m \neq 0$  and the word  $zw^mz^{-1} \in C$ , say  $1 = (zw^mz^{-1})^k = zw^{mk}z^{-1}$  in  $G$  for some  $k \neq 0$ , then  $w^{mk} = 1$  in  $G$  and  $w \in C$ . This establishes condition C2. Finally, if  $w \in C$ , then  $w^n = 1$  in  $G$ , so that  $uw^n = u$  in  $G$ , and condition C3 follows. Thus,  $C$  is a partial-Burnside set.

By our choice of  $C$ , the relators in  $\langle S | \mathfrak{R} \rangle$  all have the form  $W^n = 1$  for some word  $W \in C$ . However, if  $W \in C$ , it follows that  $W^n = 1$  is a relation in  $B_C(S, n)$ . Thus, the identity on  $S$  extends to an epimorphism  $G \rightarrow B_C(S, n)$ . On the other hand, every relator in the presentation defining  $B_C(S, n)$  is of the form  $W^n = 1$  for some word  $W \in C$ . By our choice of  $C$ , it follows that  $W$  has finite order in  $G$ ; since  $G$  is partial-Burnside of partial exponent  $n$ , it follows that  $W^n = 1$  in  $G$ . Thus, the identity map on  $S$  also extends to an epimorphism  $B_C(S, n) \rightarrow G$ . Thus,  $G \cong B_C(S, n)$ . □

In particular, a partial-Burnside group  $G$  has a graded presentation  $G_C(\infty)$  - we will say that  $G_C(\infty)$  is a minimal partial-Burnside presentation (mpbp) of  $G$ . Now, we will show that partial-Burnside groups of partial-exponent  $n$  are the 'freest' groups in which every torsion element satisfies  $x^n = 1$ .

*Remark III.5.* Let  $G = \langle S \rangle$  be a group in which every torsion element satisfies the equation  $x^n = 1$ . Let  $C$  be the set of words over  $S$  representing torsion elements of  $G$ . Then  $G$  is a canonical homomorphic image of the group  $B_C(S, n)$ .

*Proof.* First, we check that  $C$  is partial-Burnside. Condition C1 is automatic. If  $zw^mz^{-1} \in C$  for some  $m \neq 0$ , then  $1 = (zw^mz^{-1})^n = zw^{mn}z^{-1}$  so that  $1 = w^{mn}$ , and  $w \in C$ ; condition C2 is satisfied. Now, if  $w \in C$ , then  $w^n = 1$ , and it follows that  $w^nu = u$ ; condition C3 is satisfied. Therefore,  $C$  is partial-Burnside.

Note that  $B_C(S, n)$  has a presentation in which every relator is the  $n$ -th power of an element of  $C$ . By hypothesis, all  $n$ -th powers of elements of  $C$  are equal to 1 in  $G$ . It follows from Dyck's theorem that the identity function on  $S$  extends to a surjective homomorphism from  $B_C(S, n)$  to  $G$ .  $\square$

In the remainder of this section, we will show that finite subgroups, abelian subgroups, and centralizers of nontrivial elements are cyclic - just as in the case of free Burnside groups. With some modifications to allow for the possibility of elements of infinite order, proofs very similar to those of chapter 6 of [O91] can be given. This is the course we take; the necessary modifications are included below.

**Lemma III.6.** *Every subgroup of  $B_C(S, n)$  has a maximal cyclic subgroup.*

*Proof.* Let  $H$  be a subgroup of  $B_C(S, n)$ . Suppose first that  $H$  contains a nontrivial element  $X$  of finite order. By proposition III.2, no finite cyclic subgroup of  $H < B_C(S, n)$  can have order greater than  $n$ . Since no infinite cyclic group can contain a nontrivial finite cyclic group, it follows that  $H$  contains a maximal cyclic subgroup.

Now, suppose that every nontrivial element of  $H$  has infinite order. Then  $H$  contains a word  $X$  of infinite order which is conjugate to a word  $X'$  such that no nontrivial element of  $H$  is conjugate to a shorter word. Suppose there is a word  $Y \in H$  such that  $\langle Y \rangle$  properly contains  $\langle X \rangle$ . Then we must have  $Y^k = X$  in  $B_C(S, n)$  for some  $k$  with  $|k| \geq 2$ . So,  $Y^k =^j X$  for some  $j$  with  $X =^j UX'U^{-1}$ . Since  $Y$  has infinite order, it follows by lemma II.49 and the fact that periods have finite order that  $Y =^j Z(Y')^l Z^{-1}$  with  $Y'$  simple in rank  $j$ . Now, we have a reduced annular diagram  $\Delta$  of rank  $j$  for the conjugacy of  $X'$  and  $(Y')^{kl}$ . Let  $p$  and  $q$  be the contours of  $\Delta$  with  $Lab(p) \equiv (Y')^{-kl}$  and  $Lab(q) \equiv X'$ . By theorems II.56 and II.57,  $\Delta$  is an  $A$ -map and  $p$  is a smooth section of  $\partial\Delta$ . Therefore, we may apply theorem II.26 and we obtain  $|(Y')^{kl}| \leq \bar{\beta}^{-1}|X'|$ . Therefore,

$$|(Y')^l| \leq (|k|\bar{\beta})^{-1}|X'| < |X'| \quad (\text{LPP } \beta).$$

But now  $Y$  is conjugate to a word shorter than  $X'$ , contradicting the choice of  $X$ . Therefore,  $\langle X \rangle$  is a maximal cyclic subgroup of  $H$ .  $\square$

**Proposition III.7.** *Every abelian subgroup of  $B_C(S, n)$  is cyclic.*

*Proof.* Let  $H$  be an abelian subgroup of  $B_C(S, n)$ . By lemma III.6, let  $K = \langle X \rangle$  be a maximal cyclic subgroup of  $H$ . Suppose there exists  $Y \in H - K$ . Since  $H$  is abelian,  $XY = YX$  in  $B_C(S, n)$ , and therefore  $XY =^j YX$  for some  $j$ . By lemma II.50, there is some  $Z \in B_C(S, n)$  such that  $X =^j Z^k$  and  $Y =^j Z^l$ . But then the cyclic group  $\langle Z \rangle \cap H$  properly contains  $K$ , contradicting its maximality. Hence,  $H = K$  is cyclic.  $\square$

**Proposition III.8.** 1) A period of any rank has order  $n$  in  $B_C(S, n)$ .

2) If nontrivial powers  $A^k, B^l$  of periods  $A, B$  of ranks  $i, j$  are conjugate in  $B_C(S, n)$ , then  $A \equiv B$  and  $k \equiv l \pmod{n}$ .

3) If cyclic subgroups  $\langle X \rangle$  and  $\langle Y \rangle$  have nontrivial intersection, then  $\langle X \rangle \subset \langle Z \rangle$  and  $\langle Y \rangle \subset \langle Z \rangle$  for some  $Z$ .

*Proof.* The proofs of claims 1 and 2 and of the case of claim 3 when  $\langle X \rangle$  has finite order are the same as the proof of theorem 19.4 in [O91]. Therefore, we will prove only claim 3 in the case when  $\langle X \rangle$  has infinite order.

Since  $\langle X \rangle$  and  $\langle Y \rangle$  intersect nontrivially, we may write  $X^a = Y^b \neq 1$  for some  $a$  and  $b$ . Without loss of generality, we may assume  $a, b > 0$  (simply choose a different generator of  $\langle X \rangle$  or  $\langle Y \rangle$  if necessary).

Since  $X$  has infinite order, so does  $Y$ . Therefore,  $X$  and  $Y$  are not conjugate to powers of periods. So,  $X = Z_0 U^k Z_0^{-1}$  and  $Y = Z_1 V^l Z_1^{-1}$  in  $B_C(S, n)$  for some shortest possible words  $U$  and  $V$ , which are simple in every rank. Without loss of generality, we may assume  $k, l > 0$ . Now, we have

$$U^{ak} = Z_0^{-1} X^a Z_0 = Z_0^{-1} Y^b Z_0 = (Z_0^{-1} Z_1) V^{bl} (Z_0^{-1} Z_1)^{-1}$$

in  $B_C(S, n)$  and therefore in some rank, say  $j$ . Then

$$U^{tak} =^j (Z_0^{-1} Z_1) V^{tbl} (Z_0^{-1} Z_1)^{-1} \quad (\text{III.1})$$

for all  $t$ . Now, let  $\Delta$  be a reduced circular diagram for this equality. Write  $\partial\Delta = p_1 q_1 p_2 q_2$  where  $\text{Lab}(p_1) \equiv Z_0^{-1} Z_1$ ,  $\text{Lab}(q_1) \equiv V^{tbl}$ ,  $\text{Lab}(p_2) \equiv Z_1^{-1} Z_0$ , and  $\text{Lab}(q_2) \equiv U^{-tak}$ . Set  $M = \gamma^{-1}(|p_1| + |p_2| + \min(|U|, |V|))$  and choose  $t$  so large that  $|q_i| - M > n \max(|U|, |V|)$  for  $i = 1, 2$ .

Now, by theorems II.56 and II.57,  $\Delta$  is an  $A$ -map and the sections  $q_1$  and  $q_2$  are smooth of ranks  $|V|$  and  $|U|$ , respectively. By lemma II.31,  $\Delta$  has a subdiagram  $\Delta'$  with contour  $p'_1 q'_1 p'_2 q'_2$  with  $q'_i$  a subpath of  $q_i$ ,  $|p'_i| < \alpha \min(|U|, |V|)$ , and  $|q'_i| \geq n \max(|U|, |V|)$ . By lemma II.55,  $U$  is conjugate in rank  $j$  to  $V^{\pm 1}$ ; write  $U =^j ZV^{\pm 1}Z^{-1}$ . Then, using (III.1), we have  $U^{tbl} =^j ZV^{\pm tbl}Z^{-1} =^j Z_2 U^{\pm tak} Z_2^{-1}$ , where  $Z_2 \equiv ZZ_1^{-1}Z_0$ .

Now, we wish to show that  $U^{tbl} =^j Z_2 U^{-tak} Z_2^{-1}$  is impossible. Indeed, this would imply  $U^{(tbl)^2} =^j Z_2 U^{(-tak)(tbl)} Z_2^{-1} =^j Z_2^2 U^{(tak)^2} Z_2^{-2}$ . Taking  $t$  sufficiently large,  $Z_2^2 =^j U^y$  for some  $y$ , by lemma II.52. Thus,  $U^y = Z_2^2 = Z_2 (Z_2^2) Z_2^{-1} = Z_2 U^y Z_2^{-1}$ . It follows that  $Z_2 U^{-ytak} Z_2^{-1} = U^{-ytak}$ . However, from our assumption at the beginning of this paragraph,  $Z_2 U^{-ytak} Z_2^{-1} = U^{ytbl}$ . Combining these last two equalities,  $U^{yt(bl+ak)} = 1$ . Since,  $t, b, l, a, k > 0$  and  $U$  has infinite order, it must follow that  $y = 0$ , i.e.  $Z_2^2 = 1$ . Since  $n$  is odd, no element of  $B_C(S, n)$  has even order; it follows that  $Z_2 = 1$ . The contradiction  $U^{tbl} = U^{-tak}$  follows immediately.

Therefore,  $U^{tbl} =^j Z_2 U^{tak} Z_2^{-1}$ . Choosing  $t$  large enough, lemma II.52 implies that  $Z_2$  is equal in rank  $j$



to a power of  $U$ , say  $Z_2 = {}^j U^p$ . Then  $Z_0 = {}^j Z_1 Z^{-1} U^p$ . It follows that  $X = Z_0 U^k Z_0^{-1} = Z_1 Z^{-1} U^k Z Z_1^{-1} = Z_1 V^{\pm k} Z_1^{-1}$  in  $B_C(S, n)$ . In particular,  $X, Y \in \langle Z_1 V Z_1^{-1} \rangle$ .  $\square$

**Proposition III.9.** *If a word  $X$  representing a nontrivial element of  $B_C(S, n)$  is conjugate in  $B_C(S, n)$  to a power of a period  $U$ , then  $|U| \leq |X|$ .*

*Proof.* Since the period  $U$  has order  $n$  by theorem III.8, we may assume that  $X = Z U^k Z^{-1}$  with  $1 \leq |k| < \frac{n}{2}$ .

First, suppose that  $X = Z U^{\pm 1} Z^{-1}$  in  $B_C(S, n)$ . Then  $U = Z^{-1} X^{\pm 1} Z$ . Since  $U \in C$ , it follows from lemma III.1 that  $X \in C$ . If  $|X| < |U|$ , then  $X$  must be conjugate in rank  $|X| - 1$  to a period of rank  $\leq |X|$ . However, this means the period  $U$  is conjugate to a power of a different period, contradicting theorem III.8.

On the other hand, suppose  $X = Z U^k Z^{-1}$  in  $B_C(S, n)$  with  $1 < |k| < \frac{n}{2}$ . By theorem II.41, there is a reduced annular diagram  $\Delta$  with contours  $p$  and  $q$  labelled by  $X$  and  $U^{-k}$ , respectively. By theorem II.56,  $\Delta$  is an  $A$ -map. Applying lemma II.44 if necessary (altering the exponent  $k$  by a multiple of  $n$  in the process, but keeping the name  $q$  for the contour nonetheless), we may apply theorem II.57 to see that  $q$  is smooth. By theorem II.26,

$$|U| \leq \bar{\beta} |k| |U| = \bar{\beta} |q| \leq |p| = |X|.$$

$\square$

**Proposition III.10.** *The centralizer  $C_X$  of a non-trivial element  $X$  in  $B_C(S, n)$  is cyclic. Moreover,  $C_X$  is infinite if  $X$  has infinite order;  $C_X$  has order  $n$  if  $X$  is torsion.*

*Proof.* By lemma III.6,  $C_X$  has a maximal cyclic subgroup  $K = \langle V \rangle$ . First, we claim that  $X \in K$ . Indeed, since  $V \in C_X$ , we have  $VX = {}^j XV$  for some  $j$ , and it follows from lemma II.50 that  $X = Z^k$  and  $V = Z^l$  for some  $Z$ . By maximality of  $K$ , we must have  $\langle Z \rangle = K$ , i.e.  $X \in K$ . Now, suppose there is some  $Y \in C_X - K$ . Then  $XY = {}^i YX$  for some  $i$ . By lemma II.50,  $X$  and  $Y$  lie in a common cyclic subgroup of  $B_C(S, n)$ . Therefore, the cyclic subgroups  $\langle V \rangle$  and  $\langle Y \rangle$  intersect nontrivially. By proposition III.8, there is a cyclic subgroup  $\langle T \rangle$  of  $B_C(S, n)$  containing both  $K$  and  $\langle Y \rangle$ . Then  $\langle T \rangle \cap C_X$  is a cyclic group properly containing  $K$ , contradicting the maximality of  $K$ . This shows that  $C_X$  is cyclic.

Since  $X \in C_X$ , it is immediate that  $C_X$  is infinite if  $\langle X \rangle$  is. On the other hand, if  $X$  has finite order, then  $X = U A^k U^{-1}$  in  $B_C(S, n)$  with  $A$  a period of some rank. Then it is immediate that  $U A U^{-1} \in C_X$ . It is also immediate from proposition III.8 that  $U A U^{-1}$  has order  $n$ . Now, since the cyclic group  $C_X$  contains a torsion element, it has order at most  $n$ . So,  $C_X = \langle U A U^{-1} \rangle$  has order  $n$ .  $\square$

Before addressing finite subgroups of  $B_C(S, n)$ , we wish to recall a basic result. This can be found, for instance, in corollary 6.1 of [O91].

**Lemma III.11.** *Every finite non-abelian group has a non-abelian metabelian subgroup.*

**Proposition III.12.** *Every finite subgroup of  $B_C(S, n)$  is cyclic (with order dividing  $n$ ).*

*Proof.* Let  $H < B_C(S, n)$  be finite. By proposition III.7, it is sufficient to show that  $H$  is abelian. If  $H$  is not abelian, it has a nonabelian metabelian subgroup  $K$  by lemma III.11. Then  $K'$  is abelian and  $K$  has a maximal normal abelian subgroup  $L$  containing  $K'$ . So,  $K/L$  is abelian. Moreover,  $L$  is cyclic since it is abelian, by III.7. Since  $L > K' \neq 1$ , we may write  $L = \langle \ell^a \rangle$  where  $\ell$  is a period of some rank. By normality of  $L$  in  $K$ , for each  $k \in K$  there is an integer  $b_k$  such that  $k\ell^a k^{-1} = \ell^{ab_k}$ . By proposition III.8, we must have  $k\ell^a k^{-1} = \ell^a$ ; thus  $L$  is central in  $K$ . Therefore,  $K < C_{\ell^a}$ , and  $K$  is cyclic by proposition III.10; contradiction.  $\square$

## CHAPTER IV

### GENERATORS IN CONTOUR LABELS

In this section, we establish some simple but very important lemmas. Theorems I.6 and I.7 and proposition I.10 for infinite  $S$  all rely on the results of this section.

**Lemma IV.1.** *Suppose that  $\Delta$  is a reduced annular diagram over a minimal partial-Burnside presentation (MPBP) of  $B_C(S, n)$  with contours  $p$  and  $q$ , where  $p$  is smooth. If the letter  $s \in S^{\pm 1}$  appears in  $\text{Lab}(p)$ , then  $s^{\pm 1}$  appears in  $\text{Lab}(q)$ .*

*Proof.* The result follows by induction on the number of  $R$ -cells of  $\Delta$ . If  $\Delta$  has no  $R$ -cells, then  $\text{Lab}(p)$  is freely equal to a cyclic shift of  $\text{Lab}(q)$ . However, the smooth section  $p$  must have cyclically reduced label. This establishes the result in rank 0.

Now, suppose that  $\Delta$  has an  $R$ -cell. By corollary II.24,  $\Delta$  contains a  $\gamma$ -cell  $\pi$  with disjoint contiguity submaps  $\Gamma_p$  and  $\Gamma_q$  to  $p$  and  $q$ , respectively such that  $(\pi, \Gamma_p, p) + (\pi, \Gamma_q, q) > \bar{\gamma}$ . By lemma II.21,  $(\pi, \Gamma_p, p) < \bar{\alpha}$ . Therefore,  $(\pi, \Gamma_q, q) > \bar{\gamma} - \bar{\alpha} > \varepsilon$ . It follows by theorem II.25 that  $\Delta$  contains an  $R$ -cell  $\pi_0$  and a contiguity  $\Gamma_0$  of  $\pi_0$  to  $q$  with  $r(\Gamma_0) = 0$  and  $(\pi_0, \Gamma_0, q) > \varepsilon$ . Let us write  $\partial(\pi_0, \Gamma_0, q) = vsut$ ,  $q = tt'$ , and  $\partial\pi_0 = ss'$ . Then, excising the subdiagram consisting of  $\Gamma_0$  and  $\pi_0$ , we obtain a reduced annular diagram  $\Delta'$  with contours  $p$  and  $t'u^{-1}s'v^{-1}$  and with  $r(\Delta') < r(\Delta)$ . By our induction hypothesis, every  $S$ -letter of  $\text{Lab}(p)$  occurs in  $\text{Lab}(t'u^{-1}s'v^{-1})$ . Since  $(\pi_0, \Gamma_0, q) > \varepsilon > \frac{1}{n}$ , it follows that  $\text{Lab}(s)$  contains a cyclic shift of the period of  $\pi_0$ . Therefore, every letter occurring in  $\text{Lab}(\partial\pi_0)$  (and hence in  $\text{Lab}(s')$ ) occurs in  $\text{Lab}(s)$ . Since  $r(\Gamma_0) = 0$ , it follows that  $u$  and  $v$  are 0-edges and  $\text{Lab}(s) \equiv \text{Lab}(t)^{-1}$ . Putting all this together, every  $S$ -letter of  $\text{Lab}(p)$  occurs in  $\text{Lab}(q)$ .

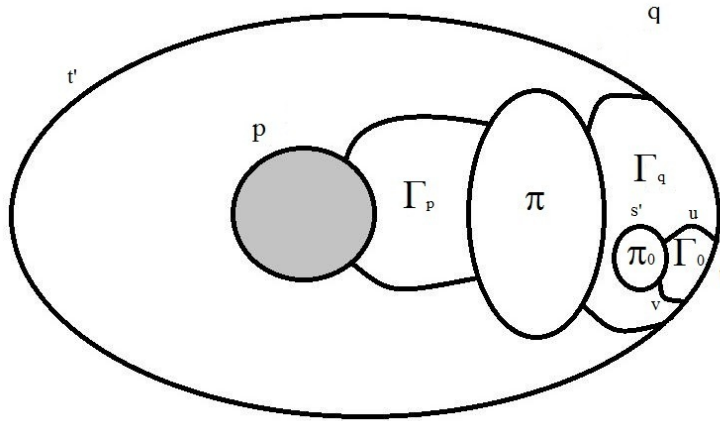


Figure IV.1: An annular diagram with contiguities illustrating the inductive step

□

**Lemma IV.2.** *Suppose  $\Delta$  is a reduced circular diagram over an MPBP of  $B_C(S, n)$ . If a letter  $s \in S^{\pm 1}$  appears in the label of an  $R$ -cell of  $\Delta$ , then  $s^{\pm 1}$  appears in the contour label of  $\Delta$ .*

*Proof.* Let  $\pi$  be an  $R$ -cell of  $\Delta$ . Excising the cell  $\pi$ , we obtain a reduced annular diagram  $\Delta'$ . By theorem II.56,  $\Delta$  is an  $A$ -map. By lemma II.17, the contour  $\partial\pi$  is smooth in  $\Delta'$ . By lemma IV.1, every  $S$ -letter in the contour label of  $\pi$  occurs in the contour label of  $\Delta$ .  $\square$

**Lemma IV.3.** *Suppose  $W$  is a word over  $S$ . Let  $U$  be a shortest word such that  $W$  is conjugate to a power of  $U$  in  $B_C(S, n)$ . If the letter  $s \in S^{\pm 1}$  appears in  $U$ , then  $s^{\pm 1}$  appears in  $W$ .*

*Proof.* Say that  $W$  is conjugate to  $U^j$ . There is a reduced annular diagram  $\Delta$  over an MPBP  $G_C(\infty)$  of  $B_C(S, n)$  with contours  $p$  and  $q$  labelled by  $U^j$  and  $W^{-1}$ , respectively.

Combining theorem III.8 with lemma III.9,  $U$  is conjugate in  $B_C(S, n)$  to a power  $V^t$  of a shortest word  $V$ , where  $V$  is either simple in every rank or a period of some rank. Now,  $W$  is conjugate  $V^{jt}$ , and it follows that  $|U| = |V|$ . In the former case, we apply theorems II.56, II.57, and II.26 to a reduced (annular) diagram for the conjugacy of  $U$  and  $V^t$ , obtaining

$$\frac{|t|}{2}|V| < \bar{\beta}|t||V| = \bar{\beta}|V^t| < |U| = |V|.$$

Hence,  $|t| = 1$ . Therefore,  $U$  is conjugate to  $V^{\pm 1}$ , and  $U$  must be simple in every rank. Applying theorems II.56 and II.57 to  $\Delta$ , the contour  $p$  is smooth.

In the latter case, we may choose  $|t| < \frac{n}{2}$  by theorem III.8. By applying lemma II.44 to a reduced diagram for the conjugacy of  $U$  and  $V^t$ , we obtain a reduced annular diagram with contour labels  $U$  and  $V^{-t+nz}$  for some integer  $z$ , in which no  $R$ -cell is  $V$ -compatible with a section of the contour. Now, we can apply theorem II.57 and theorem II.26 to this new diagram to find

$$\frac{|t|}{2}|V| < \bar{\beta}|t||V| \leq \bar{\beta}|t+nz||V| = \bar{\beta}|V^{t+nz}| < |U| = |V|.$$

It follows that  $|t| = 1$ . Thus,  $V$  is conjugate to  $U^{\pm 1}$ , and  $B_C(S, n)$  has an MPBP in which  $U$  is a period of rank  $|U|$ . Applying lemma II.44 and theorem II.57 to the diagram  $\Delta$  (modified to be a reduced diagram over the new MPBP), we find once again that  $p$  is smooth.

Therefore, in either case, the contour  $p$  is smooth. Applying lemma IV.1, every  $S$ -letter of  $U$  appears in  $W$ .  $\square$

**Lemma IV.4.** *Suppose  $\Delta$  is a circular diagram over an MPBP of  $B_C(S, n)$  with  $\partial\Delta = wv^{-1}$  and  $v$  geodesic. If a letter  $s \in S^{\pm 1}$  appears in the label of  $v$ , then  $s^{\pm 1}$  appears in the label of  $w$ .*

*Proof.* The proof follows by induction on the number of  $R$ -cells. If  $\Delta$  contains no  $R$ -cells the result is trivial, as  $Lab(v)$  is obtained from  $Lab(w)$  by free reduction.

By theorem II.56,  $\Delta$  is an  $A$ -map. If  $\Delta$  has positive rank, then  $\Delta$  contains a  $\gamma$ -cell  $\Pi$  by corollary II.24. Now,  $\Pi$  has disjoint contiguities  $\Gamma_w$  and  $\Gamma_v$  to  $w$  and  $v$  respectively such that  $(\Pi, \Gamma_w, w) + (\Pi, \Gamma_v, v) > \bar{\gamma}$ . Since  $v$  is geodesic, it follows from lemma II.21 that  $(\Pi, \Gamma_v, v) < \bar{\alpha}$ . Therefore,  $(\Pi, \Gamma_w, w) > \bar{\gamma} - (\Pi, \Gamma_v, v) > \varepsilon$ . By theorem II.25, it follows that there is an  $R$ -cell  $\Pi_0$  and a contiguity  $\Gamma$  of  $\Pi_0$  to  $w$  with  $r(\Gamma) = 0$  and  $(\Pi_0, \Gamma, w) > \varepsilon$ . Excising the subdiagram of  $\Delta$  consisting of  $\Gamma$  and  $\Pi_0$ , we obtain a reduced circular diagram  $\Delta_0$  with fewer  $R$ -cells. Write  $\partial\Delta_0 = w'v^{-1}$ . Since  $\varepsilon > \frac{1}{n}$ , every letter of  $\partial\Pi_0$  appears in  $w$ . Now, by induction hypothesis, every letter of  $Lab(u)$  appears in  $Lab(w')$  and hence in  $Lab(w)$ .  $\square$

Finally, we need a version of lemma IV.2 for annular diagrams. This will be needed to study the conjugacy problem when  $S$  is infinite.

**Lemma IV.5.** *Let  $\Delta$  be a reduced annular diagram with contours  $p$  and  $q$  over an MPBP of  $B_C(S, n)$ . If a letter  $s \in S^{\pm 1}$  appears in the label of an  $R$ -cell of  $\Delta$ , then  $s^{\pm 1}$  appears in  $Lab(p)$  or  $Lab(q)$ .*

*Proof.* Suppose that  $r(\Delta) > 0$ , for otherwise the lemma holds vacuously. By corollary II.24,  $\Delta$  has a  $\gamma$ -cell  $\Pi$ , with contiguities  $\Gamma_p$  and  $\Gamma_q$  to  $p$  and  $q$ , respectively, with  $(\Pi, \Gamma_p, p) + (\Pi, \Gamma_q, q) > \bar{\gamma}$ .

If  $\Delta$  has a single  $R$ -cell, then  $r(\Gamma_p) = 0 = r(\Gamma_q)$ . Since  $\bar{\gamma} > \frac{1}{n}$ , every letter that appears in  $Lab(\partial\Pi)$  must occur in  $Lab(\Pi \wedge p)$  or  $Lab(\Pi \wedge q)$ , and consequently in  $Lab(p)$  or  $Lab(q)$  as well.

Suppose now that  $\Delta$  has more than one  $R$ -cell. By theorem II.25,  $\Delta$  contains an  $R$ -cell  $\pi$  which is contiguous across  $\Gamma$  to  $p$  (or  $q$ ) with  $r(\Gamma) = 0$  and  $(\pi, \Gamma, p) > \varepsilon$ . Since  $\varepsilon > \frac{1}{n}$ , every letter appearing in the contour label of  $\pi$  appears in  $p$ . Excising  $\Gamma$  and  $\pi$ , we obtain a reduced annular diagram  $\Delta'$  with fewer  $R$ -cells than  $\Delta$ . It follows inductively that any letter appearing in an  $R$ -cell of  $\Delta'$  appears in  $Lab(q)$ ,  $Lab(p)$ , or  $Lab(\partial\pi)$ , and consequently in  $Lab(q)$  or  $Lab(p)$ .  $\square$

## CHAPTER V

### FREE AND FREE BURNSIDE SUBGROUPS OF PARTIAL BURNSIDE GROUPS

**Definition V.1.** For a group  $G$ , let  $Tor(G)$  be the set of elements of finite order.  $G$  is called torsion-free if  $Tor(G) = \langle 1 \rangle$ , and  $G$  is called torsion if  $Tor(G) = G$ .

Given a group  $B_C(S, n)$ , we can find noncommuting words  $T$  and  $Q$  where  $|T| < 3|Q|$ . We choose  $T$  to be geodesic and  $Q$  to be a period or a word simple in every rank. Let  $W \equiv (Q^{n_0} T Q^{n_0+2} T \dots T Q^{n_0+2h-2})^{-1}$  and

$V \equiv (Q^{n_0+1} T Q^{n_0+3} T \dots T Q^{n_0+2h-1})^{-1}$ . Now, we form a set  $\bar{S} = S \cup \{w, v\}$  and a group  $\bar{G} = \langle \bar{S} | wW^{-1}, vV^{-1}, R^n \text{ for } R \in C \rangle$ . It is worth observing now that  $\bar{G} \cong B_C(S, n)$ . Let  $\bar{C}$  denote the set of words in the alphabet  $\bar{S}$  representing elements of  $Tor(\bar{G})$ .

The proof of proposition III.4 yields the following:

*Remark V.2.* The set  $\bar{C}$  is partial-Burnside.

We now give a graded presentation for  $\bar{G}$ . Let  $L_0 = R_0 = \emptyset$ , and let  $\bar{G}(0)$  be the free group on  $\bar{S}$ . Now, suppose that  $R_i, L_i$  and  $\bar{G}(i) = \langle \bar{S} | R_i \rangle$  have been defined. Then we choose a maximal set  $L_{i+1} \subset \bar{C}$  of  $i+1$  letter words simple in rank  $i$  with the property that if  $A, B \in L_{i+1}$  and  $A$  is conjugate in rank  $i$  to  $B^{\pm 1}$ , then  $A \equiv B$ . Additionally, we choose  $L_{|Q|}$  to include  $Q$  (we will check in remark V.5 that  $Q$  is simple in rank  $|Q| - 1$ ). Now, for  $i+1 \neq |Q|$ , we define  $R_{i+1} = R_i \cup \{R^n : R \in L_{i+1}\}$ . Next, we take  $L_{|Q|}^*$  to be  $L_{|Q|}$  if  $Q \in C$  and to be  $L_{|Q|} - \{Q\}$  otherwise. We take  $R_{|Q|} = R_{|Q|-1} \cup \{R^n : R \in L_{|Q|}^*\} \cup \{wW^{-1}, vV^{-1}\}$ . We define  $\bar{G}(i+1) = \langle \bar{S} | R_{i+1} \rangle$ . We set  $\bar{G}(\infty) = \left\langle \bar{S} \mid \bigcup_{i=0}^{\infty} R_i \right\rangle$ .

The elements of  $L_i$  are periods of rank  $i$ . The relators  $R^n$  are called relators of type 1, and the relators  $wW^{-1}$  and  $vV^{-1}$  are relators of type 2. The corresponding cells are called cells of type 1 and type 2, respectively. In cells of type 2, the sections labelled by powers of  $Q$  are called long sections, and all other sections are called short.

*Remark V.3.* For  $i < |Q|$ , we can choose periods in such a way that the presentations  $G_{\bar{C}}(i)$  and  $\bar{G}(i)$  are identical. In particular, we can apply lemmas IV.1 through IV.4 for the groups  $\bar{G}(i)$  for  $i < |Q|$ .

We desire now to show that  $\bar{G}(\infty) = \bar{G}$  and that our graded presentation satisfies condition R.

**Lemma V.4.** Words  $U$  and  $V$  in the alphabet  $S$  are equal (resp. conjugate) in  $B_C(S, n)$  if they are equal (resp. conjugate) in  $\bar{G}(\infty)$ .

*Proof.* We observe that there is an epimorphism  $\bar{G}(\infty) \rightarrow \bar{G} \cong B_C(S, n)$  that fixes  $S$ . The isomorphism is clear. Now, we need only to demonstrate the existence of an epimorphism  $\bar{G}(\infty) \rightarrow \bar{G}$  fixing  $S$ . The relations in  $\bar{G}(\infty)$  consist of  $wW^{-1} = 1$ ,  $vV^{-1} = 1$ , and relations of the form  $X^n = 1$  where  $X \in \bar{C}$ . The first two relations appear in the presentation of  $\bar{G}$  as well. Now, if  $X \in \bar{C}$ , then  $X$  is equal in  $\bar{G}$  to an element of  $Tor(B_C(S, n))$ , which must have order dividing  $n$  by theorem III.8. In particular,  $X^n = 1$  in  $\bar{G}$ , and the

desired epimorphism exists.

Now, if  $U$  and  $V$  are words over  $S$  which are equal (resp. conjugate) in  $\bar{G}(\infty)$ , then  $U$  and  $V$  must have the same image (resp. conjugate images) under our homomorphism  $\bar{G}(\infty) \rightarrow B_C(S, n)$ , i.e.  $U = V$  (resp.  $U$  is conjugate to  $V$ ) in  $B_C(S, n)$ .  $\square$

*Remark V.5.* The word  $Q$  is simple in  $\bar{G}(|Q| - 1)$ .

*Proof.* Assume to the contrary that  $Q$  is conjugate in  $\bar{G}(|Q| - 1)$  to a power of a shorter word. Let  $U$  be a shortest such word. By remark V.3, we can apply lemma IV.3 to conclude that  $U$  is a word over  $S$ . By lemma V.4,  $Q$  must be conjugate in  $B_C(S, n)$  to a power of  $U$ . If  $Q$  is simple in every  $G_C(i)$ , this yields an immediate contradiction. If  $Q$  is a period in  $G_C(\infty)$ , then  $U$  must have finite order, and is therefore conjugate to a power of a period  $V$ . By lemma III.9,  $|V| \leq |U| < |Q|$ . Now, it follows that  $Q$  is conjugate to a power of  $V$ , contradicting theorem III.8.  $\square$

**Lemma V.6.** *The presentations  $\bar{G}(i)$  and  $\bar{G}(\infty)$  satisfy condition R from [O91].*

*Proof.* Conditions R1, R2, and R7 are immediate, simply requiring that  $n$  and  $n_0$  be chosen large enough. Conditions R5 and R6 are checked easily, since there are only two relations of the second type. Now, of course we have the natural epimorphisms

$$\bar{G}(|Q| - 1) \rightarrow \bar{G}(\infty) \rightarrow \bar{G} = B_C(S, n).$$

Now, if  $T \in \langle Q \rangle$  in the group  $\bar{G}(|Q| - 1)$ , then the same would hold in  $B_C(S, n)$ , and  $T$  and  $Q$  would commute here; contradiction. This establishes R4.

Recall once more that we can apply lemma IV.4 in  $\bar{G}(|Q| - 1)$ . Suppose now that  $T$  is equal in  $\bar{G}(|Q| - 1)$  to a geodesic word  $U$ . We know that  $T$  contains only letters of  $S$ . By lemma IV.4,  $U$  contains only letters from  $S$ . Then, by lemma V.4,  $T = U$  in  $B_C(S, n)$ . Therefore,  $T$  is minimal in  $\bar{G}(|Q| - 1)$  since  $T$  is geodesic in  $B_C(S, n)$ . This establishes R3.  $\square$

**Lemma V.7.**  $\bar{G}(\infty) = \bar{G}$ .

*Proof.* We showed in the proof of lemma V.4 that there is an epimorphism  $\bar{G}(\infty) \rightarrow \bar{G}$  that fixes  $S$ . Therefore, all that remains is to show that there is an epimorphism  $\bar{G} \rightarrow \bar{G}(\infty)$  fixing  $S$ .

Recall that the relations in our presentation of  $\bar{G}$  are  $wW^{-1} = 1$ ,  $vV^{-1} = 1$ , and  $X^n = 1$  for  $X \in C$ . The first two of these are also relations in  $\bar{G}(\infty)$ . Now, suppose  $X \in C$  and write  $|X| = i$ . If  $X$  is simple in  $\bar{G}(i - 1)$ , then  $X$  is conjugate in rank  $\bar{G}(i - 1)$  to a period (or inverse of a period) of the first type of rank  $i$ , and therefore  $X^n = 1$  in  $\bar{G}(i)$ . On the other hand,  $X$  could be conjugate in  $\bar{G}(i - 1)$  to a power of a shorter word; let  $U$  be a shortest such word. Since there is an epimorphism  $\bar{G}(i - 1) \rightarrow \bar{G}$ , it follows that  $X$  is conjugate to a power of  $U$  in  $\bar{G}$ . Since  $X$  has finite order in  $\bar{G}$ , so does  $U$ ; hence  $U \in \bar{C}$ . Finally,  $X^n = 1$  in  $\bar{G}(i - 1)$  since  $U^n = 1$  in  $\bar{G}(|U|)$ . Thus, the desired epimorphism exists.  $\square$

**Lemma V.8.** *Let  $\Delta$  be a reduced circular diagram over  $\tilde{G}(\infty)$ . If no  $S$ -letter appears in the contour label of  $\Delta$ , then no  $S$ -letter appears in the contour label of any  $R$ -cell.*

*Proof.* Suppose to the contrary that there is a reduced circular diagram whose contour contains no  $S$ -letter, while the label of some  $R$ -cell contains an  $S$ -letter. Let  $\Delta'$  be such a diagram having the minimum number of  $R$ -cells. By theorem II.58,  $\Delta'$  is a  $B$ -map. By corollary II.24,  $\Delta'$  has a  $\gamma$ -cell  $\Pi$  which is contiguous across  $\Gamma$  to  $\partial\Delta'$  with  $(\Pi, \Gamma, \partial\Delta') > \bar{\gamma}$ . Therefore, by theorem II.25, there is a cell  $\pi$  having a long section  $p$  which is contiguous across a subdiagram  $\Gamma_0$  to  $\partial\Delta'$  with  $(p, \Gamma_0, \partial\Delta') > \varepsilon$  and  $r(\Gamma_0) = 0$ . First, note that  $\pi$  cannot be a cell of type 2, for every long section is labelled by a word consisting only of  $S$ -letters. Consequently,  $\pi$  has type 1, and  $p = \partial\pi$ . Since  $\varepsilon > \frac{1}{n}$ , every letter that appears in  $\text{Lab}(\partial\pi)$  also appears in  $\text{Lab}(\partial\pi \wedge \Gamma_0)$  and hence in  $\text{Lab}(\partial\Delta')$  as well. In particular,  $\text{Lab}(\partial\pi)$  cannot contain any  $S$ -letters. Excising  $\pi$  and  $\Gamma_0$ , we obtain a reduced circular diagram with fewer  $R$ -cells than  $\Delta'$  whose contour label contains no  $S$ -letters, but in which there is some  $R$ -cell whose contour label contains an  $S$ -letter. This contradicts our minimality assumption about  $\Delta'$ .  $\square$

We are almost ready to prove one of our main results about subgroups of partial-Burnside groups. First, we recall:

**Lemma V.9.** *[Lemma 27.3, [O91]] Suppose that  $H$  is a noncyclic subgroup of  $B_C(S, n)$ . Then some conjugate of  $H$  contains noncommuting words  $Q$  and  $T$  where  $|T| < 3|Q|$  and  $Q$  is a period or is simple in every rank.*

**Theorem I.5.** *Let  $H$  be a noncyclic subgroup of a partial-Burnside group  $G$  of large, odd partial-exponent  $n$ . Then  $H$  contains a noncyclic partial-Burnside group of partial-exponent  $n$ .*

*Proof.* Let  $Q$  and  $T$  be noncommuting words as in V.9. Now, we make use of our presentation  $\tilde{G}(\infty)$  for the partial-Burnside group  $G = B_C(S, n)$ . We claim that  $w$  and  $v$  generate a partial-Burnside group lying in a conjugate of  $H$ . Since  $G$  is partial-Burnside, every torsion element has order dividing  $n$ ; consequently, every torsion element of  $\langle v, w \rangle$  has order dividing  $n$ .

Suppose that a word  $X(v, w) = 1$ . Then there is a reduced circular diagram  $\Delta$  over  $\tilde{G}(\infty)$  with contour label  $X(v, w)$ . By lemma V.8, no  $R$ -cell contains an  $S$ -letter in its contour label. Therefore, every  $R$ -cell of  $\Delta$  is labelled by a word of the form  $Y(v, w)^n$ . Hence,  $X(v, w) = 1$  is a consequence of  $n$ -th power relations, and  $\langle v, w \rangle$  is partial-Burnside (of partial-exponent  $n$ ). By lemma V.9,  $\langle v, w \rangle$  lies in a conjugate of  $H$ .  $\square$

**Corollary I.6.** *Every noncyclic torsion subgroup of a partial-Burnside group of sufficiently large, odd partial-exponent  $n$  contains a noncyclic free Burnside group of exponent  $n$ .*

**Corollary I.7.** *Every noncyclic torsion-free subgroup of a partial-Burnside group of sufficiently large odd partial exponent contains a noncyclic free subgroup.*

**Proposition V.10.** *Let  $C$  be a partial-Burnside set over  $S$  and  $p : B_C(S, n) \rightarrow B(S, n)$  be the natural projection. If  $H < B(S, n)$  is a noncyclic subgroup in which no nontrivial element is represented by a power of a word of  $C$ , then  $p^{-1}(H) < B_C(S, n)$  contains a noncyclic free group.*



*Proof.* By theorem I.7, it suffices to show that  $p^{-1}(H)$  is torsion-free. Let  $A \in F$  be a word representing a nontrivial element of  $B_C(S, n)$ . If  $A$  has finite order, then  $A = ZB^kZ^{-1}$  in  $B_C(S, n)$  for some period  $B$  and some integer  $k$  by theorem III.8. Since  $Z^{-1}(ZBZ^{-1})Z = B \in C$ , it follows from lemma III.1 that  $ZBZ^{-1} \in C$ . The word  $A$  can be represented in  $B_C(S, n)$  a power of an element of  $C$ . Consequently,  $A$  can be represented in  $B(S, n)$  by a power of an element of  $C$ . Therefore,  $A$  does not represent an element of  $H$ , and  $A$  does not represent an element of  $p^{-1}(H)$ . We conclude that every nontrivial element of  $p^{-1}(H)$  has infinite order.  $\square$

## CHAPTER VI

### PERIODS AVOIDING SOME NORMAL SUBGROUP

In theorem 39.4 of [O91], Olshanskii gives an example of a characteristic subgroup  ${}^nF$  of the free group  $F$  such that the quotient  $F/{}^nF$  contains a noncyclic free subgroup. This group was defined as follows: Let  $F = F(S)$ , the free group on  $S$ . We say that a word is congruent to 0 mod  $n$  if in this word every letter of  $S$  has exponents summing to a multiple of  $n$  (please note that this notation conflicts with that of [O91] - Olshanskii called such words congruent to 1 mod  $n$ ). Let  ${}^nF$  be the subgroup of  $F$  generated by the set of  $n$ th powers of words that are not congruent to 0 mod  $n$ . In our notation, we have  $F/{}^nF = B_C(S, n)$  where  $C$  is the set of words not congruent to 0 mod  $n$ . The reader should note that  ${}^nF$  is a characteristic subgroup of  $F$  that is not verbal; in fact, Olshanskii showed that  $F/{}^nF$  contains a noncyclic free subgroup. We will extend this result, showing that when  $n$  is a large prime, every noncyclic subgroup of  $F/{}^nF$  contains a noncyclic free subgroup. To help motivate our discussion, note that  $C$  avoids the normal subgroup  $F^n[F, F] > F^n$  of  $F$ .

Let  $N$  be a normal subgroup of  $F = F(S)$  containing  $F^n$ . Let  $M$  be the normal subgroup of  $F$  generated by  $\{T^n : T \notin N\}$ . In order to investigate  $F/M$ , we will first recognize it as a partial-Burnside group.

**Lemma VI.1.** *The set  $F - N$  is a partial-Burnside set, and  $F/M \cong B_{F-N}(S, n)$ .*

*Proof.* Condition C1 is completely trivial: if  $x = y$  in  $F$ , then  $x \in N \Leftrightarrow y \in N$ . Condition C2 is similarly simple:  $x \in N \Rightarrow zx^kz^{-1} \in N$ , therefore  $zx^kz^{-1} \notin N \Rightarrow x \notin N$ . To establish condition C3, simply observe that  $w^n \in F^n \subset N$  for every  $w$ . Therefore,  $u \in N \Leftrightarrow uw^n \in N$ . This shows that  $F - N$  is a partial-Burnside set. It is immediate from the definitions of  $B_{F-N}(S, n)$  and  $F/M$  that these groups are isomorphic, as they are given by the same presentation.  $\square$

Henceforth, we will write  $F/M = B_{F-N}(S, n)$  since there is a natural isomorphism fixing the generating set  $S$ . It is clear that every element of  $F - N$  represents an element of  $F/M$  whose order divides  $n$ .

**Lemma VI.2.** *The group  $N/M$  does not contain an element of order  $n$ .*

*Proof.* Suppose  $X \in N/M$  has order  $n$ . Of course,  $X$  is represented by some word  $A \in N$ . By theorem III.8,  $A$  is conjugate in  $F/M = B_{F-N}(S, n)$  to a power  $B^j$  of a period  $B$ . The conjugates  $A$  and  $B^j$  must have the same order, namely  $n$ . Since  $B$  has order  $n$  by theorem III.8,  $j$  must be relatively prime to  $n$ , and  $B = A^k$  in  $F/M$  for some  $k$ . Thus,  $B$  is freely equal to  $A^k \prod V_i R_i^{\pm n} V_i^{-1}$ , where each  $R_i$  is a period. But  $R_i^n \in F^n \subset N$  and the period  $B \in F - N$ . Therefore,  $A^k \in F - N$ , so that  $A \in F - N$ , which is a contradiction.  $\square$

**Proposition VI.3.** *If  $F/M$  contains a nontrivial normal torsion subgroup, then  $N/M$  is trivial and  $N = F^n$ . Consequently,  $F/M \cong B(S, n)$ .*

*Proof.* Suppose that  $K$  is a nontrivial normal torsion subgroup of  $F/M$ . Then  $K \cap (N/M)$  is a normal torsion subgroup of  $F/M$ . By lemma VI.2,  $K \cap (N/M)$  cannot contain a noncyclic free Burnside group of exponent  $n$ . By theorem I.6,  $K \cap (N/M)$  must be cyclic.

Next, we will show that no nontrivial finite normal subgroup of  $F/M$  is cyclic. Indeed, suppose the nontrivial finite cyclic group  $\langle A \rangle$  is normal in  $F/M$ . We can write  $A = ZB^kZ^{-1}$  for some period  $B$ . By normality, for each  $V$  we have  $VB^kV^{-1} = B^{kl}$  for some  $l$  (depending on  $V$ , perhaps). By proposition III.8, it follows that  $VB^kV^{-1} = B^k$  for every  $V$ . But this implies by theorem III.10 that  $F/M$  is cyclic; contradiction. In particular, we conclude now that  $K$  is noncyclic and that  $K \cap (N/M) = \langle 1 \rangle$ .

Since normal subgroups  $K$  and  $N/M$  intersect trivially,  $G$  contains  $K \times (N/M)$ . The centralizer of each element of  $N/M$  therefore contains the noncyclic group  $K$ . Since centralizers of nontrivial elements are cyclic (theorem III.10), it follows that  $N/M$  is trivial. Therefore,  $N = M$ . But  $N > F^n$  by hypothesis, and  $F^n > M$  by the definition of  $M$ , so  $N = M = F^n$ . Therefore,  $F/M = F/F^n \cong B(S, n)$ .  $\square$

For each letter  $s \in S$ , let  $e_s(U)$  denote the sum of the exponents of  $s$  in the word  $U$ . We say that a word  $U$  is primitive mod  $n$  if  $\gcd_{s \in S} e_s(U)$  is relatively prime to  $n$ .

**Lemma VI.4.** *Let  $H < B_C(S, n)$  be noncyclic and let  $P$  be a normal subgroup of  $B_C(S, n)$ . Suppose  $H \cap P$  contains an element  $A^k \neq 1$  where  $A$  is a period or is simple in every rank. Then  $H \cap P$  contains a conjugate of a nontrivial power  $B^l$  of an element  $B$ , where  $|B| > |A|$ , and  $B$  is a period or is simple in every rank.*

*Proof.* Of course if  $A$  is a period, then  $\langle A^k \rangle = \langle A^{n/j} \rangle$  for some odd number  $j > 1$ . In particular,  $A^{n(j-1)/2j} \in H \cap P$ , and  $\frac{n}{3} \leq \frac{n(j-1)}{2j} < \frac{n}{2}$ . Thus, we may assume  $\frac{n}{3} \leq k < \frac{n}{2}$ . On the other hand, if  $A$  has infinite order, then  $H \cap P$  will contain arbitrarily large powers of  $A$ . Thus, in either case, we may assume  $k \geq \frac{n}{3}$ .

Since  $H$  is not cyclic, it follows from lemma III.10 that there is a word  $V \in H$  that does not commute with  $A^k$ . Now, by theorem III.8,  $[A^k, V] \in H \cap P$  is conjugate in  $B_C(S, n)$  to some  $B^l$ , where  $B$  is a period of  $B_C(S, n)$  and  $0 < l < n$  or  $B$  has infinite order and is simple in every rank. Now, this conjugacy has a reduced diagram on an annulus with contours  $p_0$  and  $q_0$  labelled by  $B^l$  and  $[A^k, V]^{-1} = VA^kV^{-1}A^{-k}$ , respectively. Decompose  $q_0 = t_1q_1t_2q_2$ , where  $\text{Lab}(t_1) \equiv V \equiv \text{Lab}(t_2)^{-1}$  and  $\text{Lab}(q_1) \equiv A^k \equiv \text{Lab}(q_2)^{-1}$ . Identifying sections  $t_1$  and  $t_2^{-1}$ , we obtain a diagram on a sphere with three holes, which we may reduce to form a reduced diagram  $\Delta_0$  (of some rank) on a sphere with three holes, having contours with  $p, q, r$  with labels  $A^k, A^{-k}$ , and  $B^l$ , respectively.

Now, if  $A$  or  $B$  is a period of rank not exceeding  $r(\Delta_0)$ , we apply lemma II.44 and obtain a reduced diagram  $\Delta$  with contours (abusing notation)  $p, q, r$  with labels  $A^u, A^{-u+xn}$ , and  $B^v$  where no cells are  $A$ -compatible with  $p$  or  $q$ , no cells are  $B$ -compatible with  $r$ ,  $u \equiv k$  modulo  $n$ , and  $v \equiv l$  modulo  $n$ . If  $A$  is a period of rank greater than  $r(\Delta_0)$ , then  $A$  is simple in rank  $r(\Delta_0)$ , and we simply take  $u = k$  and  $x = 0$ . Similarly for  $B$ . By theorems II.56 and II.57,  $\Delta$  is an  $A$ -map with  $p, q$ , and  $r$  smooth sections of ranks  $|A|$ ,  $|A|$ , and  $|B|$ , respectively. Now, we note that  $|u|, |-u+xn| \geq \frac{n}{3} > 100\zeta^{-1}$ . In the case when  $A$  is a period of rank  $\leq r(\Delta_0)$ , this uses the observation that both  $u$  and  $-u+xn$  are between  $\frac{n}{3}$  and  $\frac{2n}{3}$  modulo  $n$ . In the other case, it simply follows since we have  $u > \frac{n}{3}$ , and  $x = 0$ .

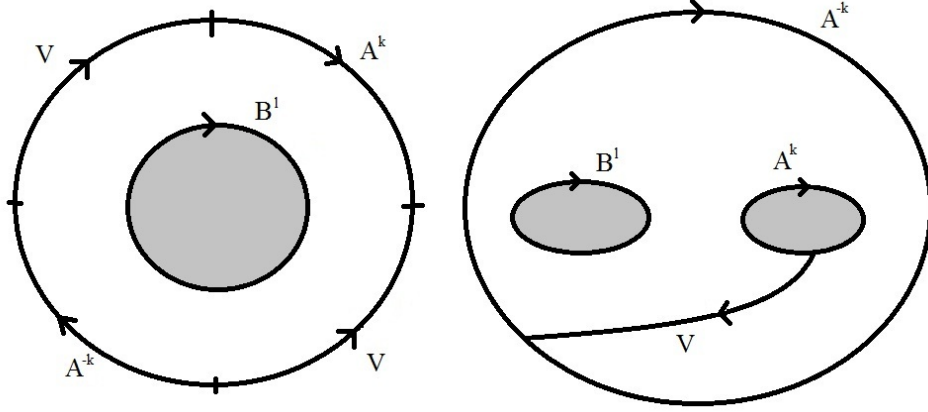


Figure VI.1: Converting the annular diagram to a diagram on a sphere with three holes

Unless  $A \equiv B$ , we have  $|v| < 100\zeta^{-1}$  by lemma II.60. Now, lemma II.61 guarantees that  $100\zeta^{-1}|B| > |B^v| \geq \zeta|A^u| \geq \frac{\zeta n}{3}|A|$ , so that  $|B| > \frac{\zeta^2 n}{300}|A| > |A|$  (LPP  $n^{-1} < \zeta$ ).

On the other hand, if  $A \equiv B$ , lemma II.61 says we must have  $|v| \cdot |A| = |A^v| > \zeta|A^u| > \frac{\zeta n}{3}|A|$ , from which it follows that  $|v| \geq \frac{\zeta n}{3} > 100\zeta^{-1}$ . It follows that  $\Delta$  is a  $G$ -map and by lemma II.34, there is a contiguity  $\Gamma$  joining two of the contours  $\bar{q}_1$  and  $\bar{q}_2$  (or one contour to itself) with degree at least  $\frac{1}{100}$ . Using our standard decomposition of contiguity submaps, we write  $\partial(\bar{q}_1, \Gamma, \bar{q}_2) = w_1 z_1 w_2 z_2$ . By lemma II.32,  $|w_1|, |w_2| < 2h\epsilon^{-1}|A| < \zeta n|A|$ . Of course, the submap  $\Gamma$  of an  $A$ -map is also an  $A$ -map by lemma II.17. Now, by lemma II.30,  $r(\Gamma) < |A|$ . If  $\text{Lab}(z_1)$  and  $\text{Lab}(z_2)$  both have period  $A$ , or both have period  $A^{-1}$ , we contradict lemma II.63, since  $|z_1| \geq \frac{|\bar{q}_1|}{100} \geq \frac{100\zeta^{-1}|A|}{100} = \zeta^{-1}|A|$ . On the other hand, if  $\text{Lab}(z_1)$  has period  $A$  and  $\text{Lab}(z_2)$  has period  $A^{-1}$ , then we apply lemma II.62, and  $z_1$  and  $z_2$  are  $A$ -compatible. Then, cutting in  $\Delta$  along the simple path from the definition of compatibility, we arrive at an annular diagram of one of two types. One possibility is that we have an annular diagram with one contour label equal to 1 in rank  $i$  and the other contour label equal to  $A^v$ ; but this is impossible since  $A^v \neq 1$ . The other possibility is that one contour label is equal in rank  $i$  to  $A^{v \pm u}$  and the other to  $A^{\mp u}$ ; but this would imply that  $A^{v \pm u}$  is conjugate to  $A^{\pm u}$ . By theorem III.8, this would imply that  $A^v = 1$ , a contradiction.  $\square$

**Proposition VI.5.** *Suppose that  $n = p$  is prime,  $H < F/M$  is noncyclic and it intersects  $N/M$  nontrivially. Then  $H \cap (N/M)$  contains a noncyclic free group.*

*Proof.* By lemma VI.2, no element of  $H \cap (N/M) < N/M$  has order  $p$ . Since  $p$  is prime, every nontrivial element of  $H \cap (N/M)$  must therefore have infinite order. Thus, by theorem I.7, we only need to show that  $H \cap (N/M)$  is noncyclic. This follows immediately from lemma VI.4.  $\square$

**Proposition VI.6.** *Let  $C$  be a partial-Burnside set over  $S$  such that every word of  $C$  is primitive mod  $n$ . Then every noncyclic subgroup of  $B_C(S, n)$  contains a noncyclic free subgroup.*

*Proof.* Let  $U$  and  $V$  be words that do not commute in  $B_C(S, n)$ . By theorem I.7, we need only show that  $\langle U, V \rangle$  contains a noncyclic torsion-free subgroup. To accomplish this, we will construct noncommuting words of  $\langle U, V \rangle$  that are congruent to 0 mod  $n$ . We note once and for all that  $F^n[F, F]$  projects naturally onto a normal subgroup  $P$  of  $B_C(S, n)$  and that a word is congruent to 0 mod  $n$  if and only if it represents an element of  $P$ .

Suppose  $U$  and  $V$  are conjugate to nontrivial powers of words  $U_0$  and  $V_0$  that are periods or simple in every rank, with  $|U_0| \geq |V_0|$ . By lemma VI.4,  $\langle U, V \rangle$  contains a word  $U^*$  congruent to 0 mod  $n$  that is conjugate to a nontrivial power of  $U_0^*$ , where  $|U_0^*| > |U_0|$  and  $U_0^*$  is a period or is simple in every rank.

We claim that  $U^*$  does not commute with  $V$ . Indeed, if they did commute, then there would be some  $Z$  such that  $U_0^*$  and  $V_0$  are conjugates of powers of  $Z$ , by theorem III.8. We can assume that  $Z$  is a shortest such word. By the definition of simplicity together with theorem III.8 and lemma III.9, it follows that  $Z$  is simple in every rank or is a period of some rank. Therefore, we must have  $|U_0^*| = |Z| = |V_0|$ , a contradiction.

Now, we have shown that  $\langle U^*, V \rangle$  is noncyclic. By lemma VI.4,  $\langle U^*, V \rangle$  contains a word  $V^*$  congruent to 0 mod  $n$  that is conjugate to a nontrivial power of  $V_0^*$ , where  $|V_0^*| > |U_0^*|$  and  $V_0^*$  is a period or is simple in every rank. Repeating the argument of the previous paragraph,  $U^*$  and  $V^*$  do not commute.

Now, the noncyclic group  $\langle U^*, V^* \rangle$  consists of words that are congruent to 0 mod  $n$ . If such a word  $W$  has finite order, it is conjugate to a power  $A^k$  of a period  $A$ , by theorem III.8. Since  $W$  is congruent to 0 mod  $n$ , so is  $A^k$ . Therefore,

$$n|e_s(A^k) = |k|e_s(A)$$

for each  $s \in S$ . This implies that  $n|k \gcd_{s \in S} e_s(A)$ . Since periods are primitive mod  $n$ , it follows that  $n|k$ , and  $A^k = 1$ . This shows that  $\langle U^*, V^* \rangle$  is torsion-free, and so contains a noncyclic free group.  $\square$

**Corollary VI.7.** *If  $n$  is a large prime, then every noncyclic subgroup of  $F / {}^n F$  contains a noncyclic free subgroup.*

## CHAPTER VII

### HYPERBOLICITY AND LACUNARY HYPERBOLICITY

Hyperbolic groups have some very simple properties, and have consequently been useful in studying more complicated groups. Therefore, it is no surprise that we can derive interesting theorems about partial-Burnside groups by noticing that our approximations  $G_C(r)$  are hyperbolic (when  $S$  is finite). Additionally, we study conditions in which partial-Burnside groups are lacunary hyperbolic. The results of this section will prove useful when we return to the topic of subgroups of partial-Burnside groups in the next section.

Given a group  $G$  with generating set  $S$ , its Cayley graph  $\Gamma(G, S)$  with respect to the generating set  $S$  is a graph with vertex set  $G$  and edge set  $G \times S^{\pm 1}$ . The directed edge  $(g, s)$  begins at  $g$  and ends at  $gs$ . To define a metric in the Cayley graph, we begin by defining a metric on each edge. Each edge, which is homeomorphic to  $[0, 1]$ , is endowed with a metric making it isometric to  $[0, 1]$ . The distance between any two points in the Cayley graph is simply the length of the shortest path joining them. In particular, the distance between two vertices is the minimum number of edges in a path between them.

A group  $G$  with finite generating set  $S$  is called hyperbolic if there exists a number  $\delta \geq 0$  such that, given any geodesic triangle in the Cayley graph  $\Gamma(G, S)$ , each side is contained in the closed  $\delta$ -neighborhood of the union of the other two sides. Hyperbolicity is a property of the group; it does not depend on the particular finite generating set chosen.

**Lemma VII.1.** *Suppose that  $x$  is a geodesic section of a reduced circular diagram  $\Delta$  over  $G_C(r)$ . Then  $x$  is smooth of rank  $M = \lceil (1 + \gamma)^{-1}(2\zeta + 1)n^2r \rceil$ .*

*Proof.* Of course, every subpath of  $x$  is geodesic. If there is a contiguity submap  $\Gamma$  of an  $R$ -cell  $\Pi$  to  $x$ , we write  $\partial(\Pi, \Gamma, x) = s_1x_1s_2x_2$ . By lemma II.18,  $|s_1x_1s_2| < (2\zeta + 1)|\partial\Pi| \leq (2\zeta + 1)nr$ . Since  $x$  is geodesic, we must have  $|\Gamma \wedge x| = |x_2| \leq |s_1x_1s_2| < (2\zeta + 1)nr$ . It follows that  $x$  is smooth of rank  $M = \lceil (1 + \gamma)^{-1}(2\zeta + 1)nr \rceil$ .  $\square$

**Proposition VII.2.** *Suppose  $\Delta$  is a reduced circular diagram over  $G_C(r)$  with  $\partial\Delta = abc$ , where  $a, b, c$  are geodesic. Then triangle  $abc$  is  $(n^3r + \frac{1}{2})$ -thin.*

*Proof.* Let  $\Delta$  be a reduced diagram over  $G_C(r)$  with  $\partial\Delta = abc$  where  $a, b, c$  are geodesics. First, we dispense with the case when  $\Delta$  does not contain an  $R$ -cell. In this case, each vertex of an  $S$ -edge in  $\partial\Delta$  is joined by a zero bond to a vertex on another side. Since each point of  $\partial\Delta$  is within  $\frac{1}{2}$  of a vertex of an  $S$ -edge, it follows that triangle  $abc$  is  $\frac{1}{2}$ -thin, and  $\frac{1}{2} \leq n^3r + \frac{1}{2}$ . In particular, we have treated the case  $r = 0$ . Henceforth, we may assume  $r > 0$ .

Suppose now that  $|a| \leq 3rn$ . In this case, we will 0-refine  $\Delta$ , writing  $\partial\Delta = abpc$  where  $|p| = 0$ .

By lemma VII.1,  $b$  and  $c$  are smooth of rank  $M = \lceil (1 + \gamma)^{-1}(2\zeta + 1)n^2r \rceil$ . By lemma II.29, any point of  $b$  (resp.  $c$ ) can be joined to some point of  $c$  (resp.  $b$ ) by a path of length at most  $\gamma^{-1}M < n^3r$ . Of course, every

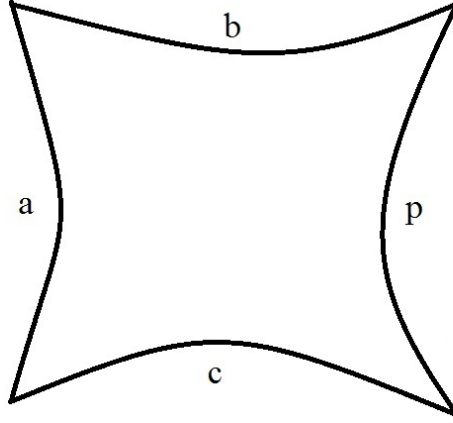


Figure VII.1: A zero-refinement of a circular diagram with three sections

point of  $a$  is joined to  $b \cup c$  by a path of length at most  $\frac{3rn}{2} < \gamma^{-1}M < n^3r$ . Thus, if  $\min(|a|, |b|, |c|) \leq 3rn$ , then  $abc$  is  $\gamma^{-1}M$ -thin, and hence  $n^3r$ -thin as well.

Henceforth, we can assume that  $|a|, |b|, |c| > 3rn$ . Now, we choose a geodesic  $x$  with  $x_-$  on  $c$  and  $x_+$  on  $a$  in such a way that  $|x| \leq rn$  and such that  $|c_1a_0|$  is maximized subject to this constraint, where  $c_1$  is the subpath of  $c$  beginning at  $x_-$  and ending at  $c_+$  and  $a_0$  is the subpath of  $a$  beginning at  $a_-$  and ending at  $x_+$ . Similarly, we choose geodesics  $y$  and  $z$  joining  $a$  to  $b$  and  $b$  to  $c$  respectively, in such a way that  $|y|, |z| \leq rn$  and  $|a_1b_0|$  and  $|b_1c_0|$  are maximized, where  $a_1$  is the subpath of  $a$  beginning at  $y_-$  and ending at  $a_+$ ,  $b_0$  is the subpath of  $b$  beginning at  $b_-$  and ending at  $y_+$ ,  $b_1$  is the subpath of  $b$  beginning at  $z_-$  and ending at  $b_+$ , and  $c_0$  is the subpath of  $c$  beginning at  $c_-$  and ending at  $z_+$ . We decompose  $a = a_0a'a_1$ ,  $b = b_0b'b_1$ , and  $c = c_0c'c_1$ .

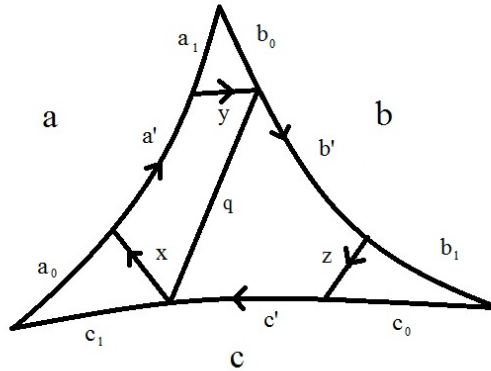


Figure VII.2: Short corner cuts in a circular diagram whose contour is a geodesic triangle

Suppose  $|a'| \leq rn$ . Let  $q$  be a geodesic joining  $y_+$  to  $x_-$ . Then  $|q| \leq 3rn$  by the triangle inequality. As

we showed two paragraphs ago, triangle  $(b'b_1)(c_0c')q^{-1}$  is  $\gamma^{-1}M$ -thin. Consequently, every point of  $b'b_1$  is within  $\gamma^{-1}M + 3rn \leq n^3r$  of  $c_0c'$ , and vice-versa. Similarly, since  $|x| \leq rn$  (resp.  $|z| \leq rn$ ), the triangle  $c_1a_0x^{-1}$  (resp.  $a_1b_0y^{-1}$ ) is  $\gamma^{-1}M$ -thin. Therefore, every point of  $a_0$  (resp.  $a_1$ ) is within  $\gamma^{-1}M + rn < n^3r$  of  $c_1$  (resp.  $b_0$ ) and vice-versa. Finally, every point of  $a'$  is clearly within  $2rn < n^3r$  of  $b$ . Thus, if  $\min(|a'|, |b'|, |c'|) \leq rn$ , then  $abc$  is  $n^3r$ -thin.

From here on, we may assume  $|a'|, |b'|, |c'| > rn$ . Let  $\Delta'$  be the diagram with contour  $a'yb'zc'x$ .

If  $\Delta'$  contains no  $R$ -cells, then each vertex of  $a'$  is connected another side of the hexagon by a zero bond. Additionally, each point of  $a'$  is within  $\frac{1}{2}$  of a vertex, and each point of  $b' \cup c' \cup x \cup y \cup z$  is within  $\max(|x|, |y|, |z|) \leq rn$  of  $b' \cup c'$ . Thus, each point of  $a'$  is within  $rn + \frac{1}{2} < n^3r$  of  $b' \cup c'$ . Similarly, each point of  $b'$  is within  $n^3r$  of  $a' \cup c'$  and each point of  $c'$  is within  $n^3r$  of  $a' \cup b'$ . Thus, if  $\Delta'$  contains no  $R$ -cells, then triangle  $abc$  is  $n^3r$ -thin.

Suppose now that  $\Delta'$  contains an  $R$ -cell. Then, by corollary II.23,  $\Delta'$  contains a  $\gamma$ -cell  $\pi$ . Consequently, there are contiguities  $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6$  of  $\pi$  to  $a', b', c', x, y, z$  (some may be absent) such that the sum of the degrees is more than  $\bar{\gamma}$ . If we ignore the contiguities  $\Gamma_i$  with degree less than  $\varepsilon$ , the degrees of the remaining contiguity submaps sum to more than  $\bar{\gamma} - 6\varepsilon > \bar{\beta}$ . Since side of the hexagon is geodesic, and therefore smooth, it follows from lemma II.21 that none of the contiguities  $\Gamma_i$  have degree exceeding  $\bar{\alpha}$ ; hence at least two of the contiguities exist.

Case 1: There are contiguities  $\Gamma_1$  and  $\Gamma_2$  of  $\pi$  to two of  $a', b'$ , and  $c'$  (say,  $a'$  and  $b'$ ) with  $(\pi, \Gamma_1, a'), (\pi, \Gamma_2, b') \geq \varepsilon$ . Write  $\partial(\pi, \Gamma_1, a') = s_1 t_1 s_2 t_2$  and  $\partial(\pi, \Gamma_2, b') = s^1 t^1 s^2 t^2$ . Write  $\partial\pi = u(t^1)^{-1} v t_1^{-1}$ .

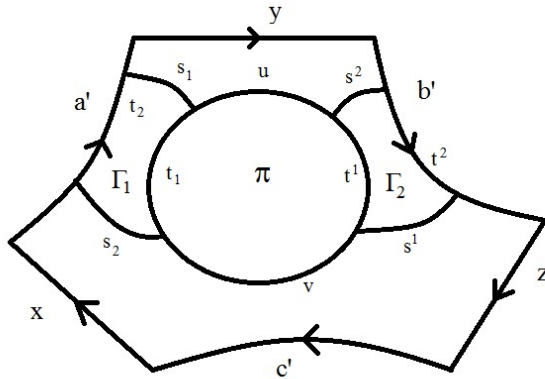


Figure VII.3: Contiguities in the hexagon to two of the triangle's sides

By lemma II.18,  $|s^1 v s_2| < (2\zeta + 1 - 2\varepsilon)|\partial\pi| < rn$ . By lemma II.19,  $|t_2| > (1 - 2\beta)|t_1| \geq (1 - 2\beta)\varepsilon|\partial\pi| > 0$ , and this contradicts our choice of  $y$ . Therefore, this case is impossible.

Case 2: There are contiguities  $\Gamma_1$  and  $\Gamma_2$  of degree at least  $\varepsilon$  to opposite sides of the hexagon, say to  $a'$



and  $z$ , respectively. Write  $\partial(\pi, \Gamma_1, a') = s_1^1 t_1^1 s_2^1 t_2^1$ ,  $\partial(\pi, \Gamma_2, z) = s_1^2 t_1^2 s_2^2 t_2^2$ ,  $\partial\pi = (t_1^1)^{-1} u (t_1^2)^{-1} v$ ,  $z = z_1 t_2^2 z_2$ , and  $a' = A_1 t_2^1 A_2$ . Let  $T_0$  be a geodesic joining  $(s_2^1)_+$  to  $(s_1^2)_-$ , let  $T_1$  be a geodesic joining  $a'_-$  to  $(s_1^2)_-$ , and let  $T_2$  be a geodesic joining  $x_-$  to  $(s_1^1)_-$ .

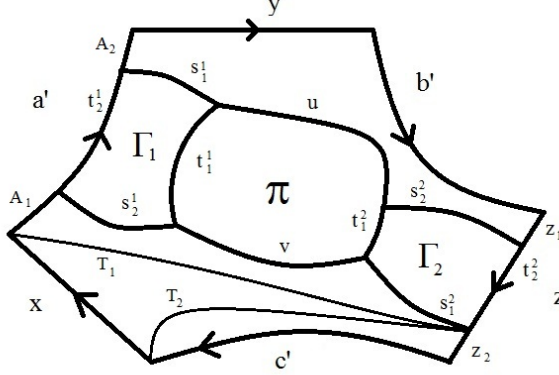


Figure VII.4: Contiguities to opposite sides of the hexagon

By the triangle inequality and lemma II.18,  $|T_0| \leq |s_1^2 v s_2^1| < (2\zeta + 1 - 2\varepsilon) < rn$ . Therefore, triangle  $A_1 T_0 T_1^{-1}$  is  $\gamma^{-1}M$ -thin, and each point of  $A_1$  (resp.  $T_1$ ) is within  $\gamma^{-1}M + rn$  of some point of  $T_1$  (resp.  $A_1$ ). Similarly,  $|x| \leq rn$  and  $|z_2| \leq rn$ , and these statements imply that triangles  $x T_1 T_2^{-1}$  and  $c' T_2 z_2$  are  $\gamma^{-1}M$ -thin. Therefore, every point of  $T_1$  (resp.  $T_2$ ) is within  $\gamma^{-1}M + rn$  of  $T_2$  (resp.  $T_1$ ), and every point of  $T_2$  (resp.  $c'$ ) is within  $\gamma^{-1}M + rn$  of  $c'$  (resp.  $T_2$ ). Putting these estimates together, every point of  $A_1$  (resp.  $c'$ ) is within  $3(\gamma^{-1}M + rn) < n^3 r$  of  $c'$  (resp.  $A_1$ ). Likewise, every point of  $A_2$  (resp.  $b'$ ) is within  $3(\gamma^{-1}M + rn) < n^3 r$  of  $b'$  (resp.  $A_2$ ).

Finally, since  $(\pi, \Gamma_1, a') < \bar{\alpha}$  by lemma II.21, it follows from lemma II.19 that  $|t_2^1| < (1 + 2\beta)|t_1^1| < (1 + 2\beta)\bar{\alpha}|\partial\pi| < rn$ . Thus, every point of  $t_2^1$  is within  $3\gamma^{-1}M + 4rn < n^3 r$  of  $c'$ . Therefore, in this case, triangle  $abc$  is  $n^3 r$ -thin.

Case 3: There are contiguities  $\Gamma_1$  and  $\Gamma_2$  of degree at least  $\varepsilon$  of  $\pi$  to two of  $x, y, z$ , say  $y$  and  $x$  respectively. Write  $\partial(\pi, \Gamma_1, y) = s_1^1 t_1^1 s_2^1 t_2^1$ ,  $\partial(\pi, \Gamma_2, x) = s_1^2 t_1^2 s_2^2 t_2^2$ ,  $\partial\pi = (t_1^1)^{-1} u (t_1^2)^{-1} v$ ,  $y = y_1 t_2^1 y_2$ , and  $x = x_1 t_2^2 x_2$ . Let  $T_1$  be a geodesic from  $x_-$  to  $y_+$ , and let  $T_2$  be a geodesic from  $z_+$  to  $y_+$ .

From the triangle inequality and lemma II.18,  $|T_1| \leq |x_1 (s_2^2)^{-1} u^{-1} (s_1^1)^{-1} y_2| < 2rn + (2\zeta + 1 - 2\varepsilon)|\partial\pi| < 3rn$ . Similarly,  $|a'| < 3rn$ . Since  $|T_1| < 3rn$ , triangle  $T_1 T_2^{-1} c'$  is  $\gamma^{-1}M$ -thin, and every point of  $c'$  (resp.  $T_2$ ) is within  $\gamma^{-1}M + 3rn$  of  $T_2$  (resp.  $c'$ ). Since  $|z| \leq rn$ , triangle  $T_2 b' z$  is  $\gamma^{-1}M$ -thin, and every point of  $T_2$  (resp.  $b'$ ) is within  $\gamma^{-1}M + rn$  of  $b'$  (resp.  $T_2$ ). Combining these estimates, every point of  $c'$  (resp.  $b'$ ) is within  $2\gamma^{-1}M + 4rn < n^3 r$  of  $b'$  (resp.  $c'$ ). Finally, since  $|a| \leq 3rn$ , it follows that every point of  $a'$  is connected to a point of  $b'$  by a path of length at most  $|a'y| \leq 4rn < n^3 r$ . Therefore, in this case, triangle  $abc$  is  $n^3 r$ -thin.

Case 4: There are contiguities  $\Gamma_1$  and  $\Gamma_2$  of degree at least  $\varepsilon$  of  $\pi$  to adjacent sides of the hexagon, say to  $a'$

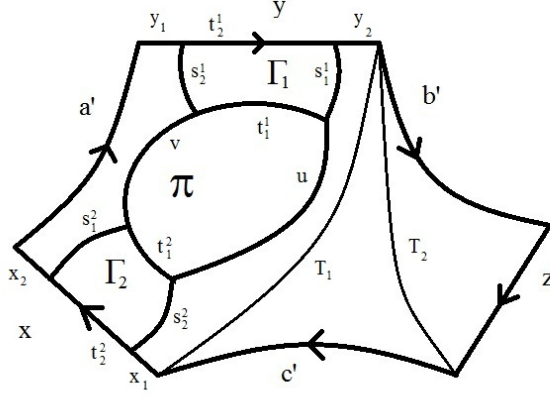


Figure VII.5: Contiguities in the hexagon to two corner cuts

and  $y$ , respectively, such that  $(\pi, \Gamma_1, a') + (\pi, \Gamma_1, y) > \bar{\beta}$ . Write  $\partial(\pi, \Gamma_1, a') = s_1^1 t_1^1 s_2^1 t_2^1$ ,  $\partial(\pi, \Gamma_2, y) = s_1^2 t_1^2 s_2^2 t_2^2$ ,  $\partial\pi = u(t_1^2)^{-1}v(t_1^1)^{-1}$ , and  $y = y_1 t_2^2 y_2$ .

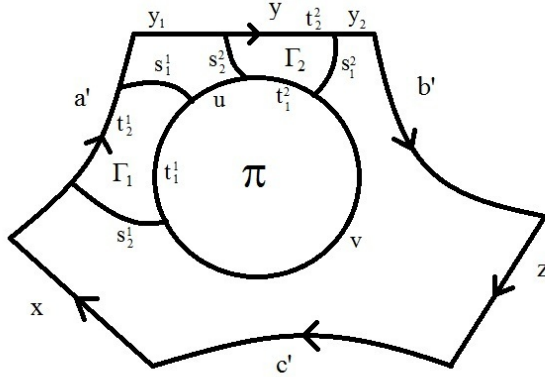


Figure VII.6: Contiguities to adjacent sides of the hexagon

By lemma II.18,

$$|(s_2^1)^{-1}v^{-1}(s_1^2)^{-1}| < (2\zeta + \beta)|\partial\pi|. \quad (\text{VII.1})$$

By lemma II.21,  $(\pi, \Gamma_1, a') < \bar{\alpha}$ , so  $(\pi, \Gamma_2, y) > \bar{\beta} - \bar{\alpha} = \frac{1}{2} - \beta - \alpha > \frac{1}{3}$ . Therefore, lemma II.18 implies that  $|t_2^2| > (1 - 2\beta)|t_1^1| > \frac{1 - 2\beta}{3}|\partial\pi|$ . Combining this with (VII.1), we see that  $|(s_2^1)^{-1}v^{-1}(s_1^2)^{-1}| < (2\zeta + \beta)\frac{3}{1 - 2\beta}|t_2^2| < 4\beta|t_2^2|$ . Therefore,  $|(s_1^1 vs_2^1)^{-1}y_2| < |y| \leq rn$ . Since  $|t_1^1| > (1 - 2\beta)|t_1^1| > 0$  by lemma II.18, this contradicts our choice of  $y$ . Therefore, this case is impossible.  $\square$

**Proposition VII.3.** *The partial-Burnside presentation  $G_C(r)$  is  $(n^3r + 1)$ -hyperbolic.*

*Proof.* Consider a geodesic triangle  $abc$  in the Cayley graph  $\Gamma(G_C(r), S)$ , where  $a, b, c$  are the sides of the triangle. If a vertex of the triangle, say  $a_+ = b_-$  is not a vertex of the Cayley graph, then there is a vertex  $V$  of the Cayley graph that is on both  $a$  and  $b$  such that  $a$  coincides with  $b^{-1}$  between  $V$  and  $a_+$ . Therefore, it suffices to consider triangles whose vertices are vertices of the Cayley graph.

Since closed paths in Cayley graphs have trivial label,  $Lab(a)Lab(b)Lab(c) = 1$  in  $G_C(r)$ . Therefore, there is a reduced circular diagram  $\Delta$  over  $G_C(r)$  whose contour label is  $Lab(a)Lab(b)Lab(c)$ . Write  $\partial\Delta = a'b'c'$  where  $Lab(a') \equiv Lab(a)$ ,  $Lab(b') \equiv Lab(b)$ , and  $Lab(c') \equiv Lab(c)$ . Since  $a, b, c$  are geodesic, so are  $a', b', c'$ .

There is a natural mapping  $\phi$  from the diagram  $\Delta$  to the Cayley graph. Let  $\phi(a'_-) = a_-$  and let  $\phi$  be a graph morphism that respects edge labels. In other words, if  $e'$  is an  $S$ -edge in  $\Delta$  its image  $\phi(e')$  will be the edge in the Cayley graph such that  $\phi(e')_- = \phi(e'_-)$  and  $Lab(\phi(e')) \equiv Lab(e')$ . It follows that  $\phi(e')_+ = \phi(e'_+)$ . Let  $\phi$  map 0-edges to vertices. Since nullhomotopic loops in  $\Delta$  have trivial contour label, and every path in the Cayley graph with trivial label is closed, it follows that  $\phi$  is well-defined. It is apparent that  $a', b', c'$  are mapped surjectively onto  $a, b, c$ , respectively.

Let  $A$  be a vertex of  $\Gamma$  on the geodesic  $a$ . Then some vertex  $A'$  of  $a'$  is mapped to  $A$  by  $\phi$ . By proposition VII.2, there is a path  $d$  in  $\Delta$  from  $A$  to  $b' \cup c'$  with  $|d| \leq n^3r + \frac{1}{2}$ . Then  $\phi(d)$  is a path in  $\Gamma$  from  $A$  to  $b \cup c$  with  $\phi(d) \leq n^3r$ . Since every point of  $\Gamma$  is within  $\frac{1}{2}$  of a vertex, it follows that every point on  $a$  is connected to  $b \cup c$  by a path of length at most  $n^3r + 1$ . The same argument works for sides  $b$  and  $c$  as well.  $\square$

This result allows us to characterize the hyperbolic partial-Burnside groups of large, odd partial-exponent.

**Theorem VII.4.** *Let  $G$  be a finitely generated partial-Burnside group of large, odd partial-exponent  $n$ . The following are equivalent:*

- (a)  $G$  is hyperbolic.
- (b)  $G$  has finitely many conjugacy classes of torsion elements.
- (c) Every minimal partial-Burnside presentation of  $G$  is finite.
- (d)  $G$  is finitely presentable.

*Proof.* (a)  $\Rightarrow$  (b): This implication is well-known. For instance, see [Gr].

(b)  $\Rightarrow$  (c): We prove the contrapositive. Suppose that  $G$  has a minimal partial-Burnside presentation  $G_C(\infty)$  with infinitely many relators. Since each relator in  $G_C(\infty)$  is the  $n$ -th power of a period, there are infinitely many periods, each of which is a torsion element. By theorem III.8, different periods are not conjugate to one another. Consequently, there are infinitely many conjugacy classes of torsion elements.

(c)  $\Rightarrow$  (d): Trivial.

(d)  $\Rightarrow$  (a): Let  $G_C(\infty)$  be a fixed minimal partial-Burnside presentation of  $G$ . If  $G$  is finitely presentable,

then it has a finite presentation consisting only of generators and relations that are present in  $G_C(\infty)$ . Consequently,  $G$  has a finite presentation of the form  $G_C(r)$ . Therefore,  $G$  is hyperbolic by proposition VII.3.  $\square$

In [OOS], Olshanskii, Osin, and Sapir consider a generalization of hyperbolic groups. In order to make the generalization clear, let us recall a characterization of hyperbolic groups. A finitely generated group is hyperbolic if and only if every asymptotic cone is an  $\mathbb{R}$ -tree. In the appendix of [OOS], Kapovich and Kleiner show that a finitely presented group is hyperbolic if and only if at least one of its asymptotic cones is an  $\mathbb{R}$ -tree.

**Definition VII.5.** A finitely generated group is called lacunary hyperbolic if and only if at least one of its asymptotic cones is an  $\mathbb{R}$ -tree.

**Definition VII.6.** Given a homomorphism  $f : G \rightarrow H$ , the injectivity radius of  $f$  with respect to the finite generating set  $S$  of  $G$ , denoted by  $r_S(f)$ , is the maximum radius of a ball in the Cayley graph  $\Gamma(G, S)$  upon which  $f$  is injective.

**Theorem VII.7.** (Theorem 3.3 [OOS]) *A finitely generated group is lacunary hyperbolic if and only if it is a direct limit of a sequence of finitely generated groups and epimorphisms*

$$G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} \dots$$

*such that each group  $G_i$  is generated by a finite set  $S_i$ ,  $\alpha(S_i) = S_{i+1}$ , and each  $G_i$  is  $\delta_i$ -hyperbolic, where  $\delta_i = o(r_{S_i}(\alpha_i))$ .*

This theorem permits an identification of lacunary hyperbolic partial-Burnside groups of large, odd partial-exponent. Essentially, a partial-Burnside group is lacunary hyperbolic if its periods are spaced sufficiently far apart.

**Theorem VII.8.** *Let  $G$  be a partial-Burnside group of large, odd partial-exponent  $n$ . Suppose  $G$  has a minimal partial-Burnside presentation  $G_C(\infty)$  with finite generating set  $S$  and the integers which lengths of periods are  $a_1 < a_2 < \dots$ . If  $a_i = o(a_{i+1})$ , then  $G$  is lacunary hyperbolic.*

*Proof.* First,  $G_C(\infty)$  is a limit of the sequence  $G_C(a_1) \rightarrow G_C(a_2) \rightarrow \dots$ . By proposition VII.3 each group  $G_C(a_i)$  is  $(n^3 a_i + 1)$ -hyperbolic.

Next, we need to find the injectivity radius of the natural projection  $G_C(a_i) \rightarrow G_C(a_{i+1})$ . Suppose that  $W = V$  in  $G_C(a_{i+1})$  but not in  $G_C(a_i)$ . Then  $WV^{-1} = 1$  in  $G_C(a_{i+1})$ . In particular, there is a reduced circular diagram  $\Delta$  over  $G_C(a_{i+1})$  with contour label  $WV^{-1}$ . Since  $WV^{-1} \neq 1$  in  $G_C(a_i)$ , there is no such diagram over  $G_C(a_i)$ . Consequently,  $\Delta$  must contain a cell of rank  $a_{i+1}$ . By corollary II.27,  $|\partial\Delta| > \bar{\beta} a_{i+1} n$ . Thus, one of  $W, V$  has length at least  $\frac{\bar{\beta} a_{i+1} n}{2} > \frac{a_{i+1} n}{3}$ . Hence, the injectivity radius is greater than  $\frac{a_{i+1} n}{3}$ . Since  $a_i = o(a_{i+1})$ , it follows that  $n^3 a_i + 1 = o\left(\frac{a_{i+1} n}{3}\right)$ . By theorem VII.7,  $G$  is lacunary hyperbolic.  $\square$

## CHAPTER VIII

### FINITELY PRESENTED SUBGROUPS OF PARTIAL-BURNSIDE GROUPS

In order to investigate finitely presented subgroups of partial-Burnside groups, it is useful to first recall a few facts about elementary groups.

Recall that a group is called elementary if it has a finite index cyclic subgroup.

**Theorem VIII.1.** *[[Gr]] Every nonelementary subgroup of a hyperbolic group contains a noncyclic free subgroup.*

**Proposition VIII.2.** *Every elementary subgroup of  $B_C(S, n)$  is cyclic.*

*Proof.* Let  $H$  be a noncyclic subgroup of  $B_C(S, n)$ . We will show that  $H$  is not elementary. Let  $A \in H$ . By theorem III.8,  $A = ZB^kZ^{-1}$  in  $B_C(S, n)$ , where  $B$  is a period of some rank or  $B$  is simple in every rank.

From here, we proceed in two cases. First, suppose that  $B$  is simple in every rank. By proposition III.10, every centralizer in  $B_C(S, n)$  is cyclic. Since  $H$  is not cyclic, it follows by theorem III.12 that there exists  $D \in H$  that does not commute with  $ZB^kZ^{-1}$ . By theorem III.8, for every integer  $l$ ,  $D$  and  $ZB^{lk}Z^{-1}$  do not commute in  $B_C(S, n)$ . We claim that the elements  $Y_q = [D, ZB^{qkn}Z^{-1}]$  for  $q = 1, 2, \dots$  are representatives of distinct cosets of  $\langle ZB^kZ^{-1} \rangle$  in  $H$ .

Indeed, the equation  $Y_{q_0} = Y_{q_1} ZB^tZ^{-1}$  is equivalent to the equation

$$[D, ZB^{(q_0 - q_1)kn}Z^{-1}] = ZB^tZ^{-1}.$$

This equation has a reduced diagram  $\Delta$  on a sphere with 3 holes with contours labeled by  $B^{(q_0 - q_1)kn}$ ,  $B^{(q_1 - q_0)kn}$ , and  $B^t$ . Since  $B$  is simple in rank  $r(\Delta)$ , it follows from theorems II.56 and II.57 that each contour is smooth of rank  $|B|$ . Since  $|(q_0 - q_1)n| \geq n > 100\zeta^{-1}$ , we have  $|t| \leq 100\zeta^{-1}$  by lemma II.60. So lemma II.61 now says  $100\zeta^{-1}|B| \geq |B^t| \geq \zeta|B^{(q_0 - q_1)kn}| \geq n\zeta|B|$ . The contradiction  $|B| > |B|$  follows quickly. Thus,  $Y_{q_0}$  and  $Y_{q_1}$  are representatives of different cosets of  $\langle ZB^kZ^{-1} \rangle$ , and  $[H : \langle A \rangle] = [H : \langle ZBZ^{-1} \rangle] = \infty$ .

The other case to consider is when  $B$  is a period of some rank. In this case,  $\langle A \rangle = \langle ZB^kZ^{-1} \rangle$  is finite by theorem III.8. By proposition III.12,  $H$  is infinite since it is not cyclic. Therefore,  $[H : \langle A \rangle] = \infty$ .

Therefore, the noncyclic group  $H < B_C(S, n)$  is not elementary. □

Since the groups  $G_C(i)$  are partial-Burnside, we immediately obtain:

**Corollary VIII.3.** *If  $S$  is finite, then every elementary subgroup of  $G_C(i) \cong B_C^i(S, n)$  is cyclic.*

**Lemma VIII.4.** *If  $H$  is a finitely generated subgroup of the partial-Burnside group  $B_C(S, n)$ , then  $H$  is a subgroup of a finitely generated partial-Burnside group of partial-exponent  $n$ .*

*Proof.* Let  $H = \langle X | \mathfrak{R} \rangle$ , with  $X$  finite. For each  $x \in X$ , we can write  $x = w_x(S)$  for some word  $w_x$ . Let  $S' = \bigcup_{x \in X} \{s \in S : s^{\pm 1} \text{ appears in } w_x\}$ . Of course  $S'$  is finite since  $X$  is finite and each  $w_x$  has only finitely many letters. By lemma IV.2, each relator  $R \in \mathfrak{R}$  is a consequence of relators of  $B_C(S, n)$  which contain only letters of  $S'$ . Once we observe that  $C' = F(S') \cap C$  is a partial-Burnside set of partial-exponent  $n$  over  $S'$ , it follows that each relator  $R \in \mathfrak{R}$  is a consequence of the relators of  $B_{C'}(S', n)$ . By Dyck's theorem, there is therefore a unique homomorphism  $H \rightarrow B_{C'}(S', n)$  taking  $x \mapsto w_x$  for each  $x \in X$ . Similarly, there is a homomorphism  $B_{C'}(S', n) \rightarrow B_C(S, n)$  fixing  $S'$ . Consequently, the composition  $H \rightarrow B_{C'}(S', n) \rightarrow B_C(S, n)$  takes  $x \mapsto w_x$  for each  $x \in X$ ; however, our composition coincides with the embedding of  $H$  into  $B_C(S, n)$ . Thus,  $H$  is embedded in  $B_{C'}(S', n)$ .  $\square$

**Theorem I.8.** *Suppose that  $H$  is a noncyclic finitely presented subgroup of a partial-Burnside group of large odd partial exponent  $n$ . Then  $H$  contains a noncyclic free group.*

*Proof.* Let  $H = \langle X, \mathfrak{R} \rangle$  be a finite presentation of  $H$ . By lemma VIII.4,  $H$  is a subgroup of a finitely generated partial-Burnside group  $B_C(S, n)$ . Now, each relator  $R \in \mathfrak{R}$  is a consequence of finitely many relators of  $B_C(S, n)$ . Consequently, for some  $i$ , every relator  $R \in \mathfrak{R}$  is a consequence of the relators of  $G_C(i)$ . Arguing as in the proof of lemma VIII.4, we see that  $H$  is a subgroup of  $G_C(i)$ .

Since  $H$  is noncyclic, corollary VIII.3 implies that  $H$  is not elementary. Since  $G_C(i)$  is hyperbolic by proposition VII.3, we therefore conclude by theorem VIII.1 that  $H$  contains a noncyclic free subgroup.  $\square$

**Corollary I.9.** *Every finitely presented subgroup of a free Burnside group of sufficiently large odd exponent is cyclic.*

## CHAPTER IX

### SOME RESULTS ON MAPS

We will now establish some results about maps that will enable us to compute space and isodiametric functions for partial-Burnside groups. These results are analogous to results from chapter 5 of [O91], where results are proved for 'narrow' maps with contour  $p_1q_1p_2q_2$  with  $q_1$  and  $q_2$  smooth. While this is very useful for contiguity diagrams between cells, we will later require similar results for contiguity diagrams to the boundary of a map; in this case, we lose smoothness of one of the sections. Therefore, we will consider 'narrow' maps with contour  $p_1q_1p_2q_2$  where  $q_2$  is smooth and  $q_1$  satisfies a weaker property of being in some sense 'reduced.' We begin by making this precise.

**Definition IX.1.** Let  $q$  be a section of the contour of a map  $\Delta$  and  $\Phi$  a positive number. Then  $q$  is  $\Phi$ -reduced in  $\Delta$  if there is no R-cell in  $\Delta$  which is contiguous to  $q$  with degree at least  $\Phi$ .

**Lemma IX.2.** Let  $\Delta$  be an A-map with contour  $p_1q_1p_2q_2$  where  $|p_1| + |p_2| < \gamma|q_2|$  where  $q_2$  is a smooth section of  $\partial\Delta$  and  $q_1$  is  $(1 - 2\beta)$ -reduced in  $\Delta$ . Then there is a 0-bond between  $q_1$  and  $q_2$  or there is an R-cell  $\Pi$  and disjoint contiguity submaps  $\Gamma_1, \Gamma_2$  from  $\Pi$  to  $q_1, q_2$  respectively, with  $(\Pi, \Gamma_1, q_1) + (\Pi, \Gamma_2, q_2) > \bar{\beta}$  (we call  $\Pi$  a  $\beta$ -cell).

*Proof.* First suppose that  $r(\Delta) = 0$ . Then each S-edge of  $\partial\Delta$  is joined to a different edge of  $\partial\Delta$  by a 0-bond. First we note that no edges of  $q_2$  are joined by a 0-bond. Indeed, if edges  $e$  and  $f$  of  $q_2$  are adjacent, we can find adjacent edges  $e'$  and  $f'$  where  $e'sf'$  is a subpath of  $q_2$  and  $|s| = 0$ . But then  $(e)_-$  and  $(f)_+$  are joined by a contiguity arc of length 0, contradicting the fact that the length 2 subpath  $e'sf'$  of the smooth section  $q_2$  is geodesic. Hence, each S-edge of  $q_2$  is joined to an edge of  $p_1q_1p_2$  by a 0-bond. Since this correspondence is injective and  $|q_2| > |p_1| + |p_2|$ , there is a 0-bond between  $q_1$  and  $q_2$ .

Now, we deal with the case  $r(\Delta) > 0$ . In this case,  $\Delta$  has a  $\gamma$ -cell  $\pi$  by corollary II.23. We proceed by induction on the number of R-cells.

Case 1: There is a contiguity submap  $\Gamma$  from  $\pi$  to  $p_1$  (or  $p_2$ ) with  $(\pi, \Gamma, p_1) > \bar{\alpha}$  (or  $(\pi, \Gamma, p_2) > \bar{\alpha}$ ). Write  $\partial(\pi, \Gamma, p_i) = s_1t_1s_2t_2$ , let  $\bar{t}$  denote the complement of  $t_1$  in  $\partial\pi$ , and write  $p_1 = ut_2v$  (or  $p_2 = ut_2v$ ).

By lemma II.20,  $|s_1\bar{t}^{-1}s_2| < |t_2|$ . Then, setting  $\bar{p}_1 = us_2^{-1}\bar{t}s_1^{-1}v$ , we have  $|\bar{p}_1| = |u| + |s_1\bar{t}^{-1}s_2| + |v| < |u| + |t_2| + |v| = |p_1|$  and we can apply the induction hypothesis to the submap  $\bar{\Delta}$  with contour  $\bar{p}_1q_1p_2q_2$  (or  $p_1q_1\bar{p}_2q_2$ ). Hence, the result holds for  $\Delta$ .

Case 2a: There are disjoint contiguity submaps  $\Gamma_p$  and  $\Gamma_q$  to  $p_1$  (or  $p_2$ ) and  $q_1$  respectively with  $(\pi, \Gamma_p, p_1) + (\pi, \Gamma_q, q_1) > \bar{\gamma}$ . Write  $\partial(\pi, \Gamma_q, q_1) = s^1t^1s^2t^2$ ,  $\partial(\pi, \Gamma_p, p_1) = s_1t_1s_2t_2$ ,  $\partial\pi = w^1t_1w_1t^1$ , and  $p_1 = \bar{v}t_2\bar{u}$ .

Since  $(\pi, \Gamma_q, q_1) < 1 - 2\beta$ , we have  $(\pi, \Gamma_p, p_1) > \bar{\gamma} - (1 - 2\beta) = 2\beta - \gamma$ . By lemma II.19,  $|t_2| > (\bar{\beta}(2\beta - \gamma) - 2\zeta)|\partial\pi|$ . Also,  $|s^1w_1^{-1}s_2| < (2\zeta + \gamma)|\partial\pi|$  by lemma II.18.

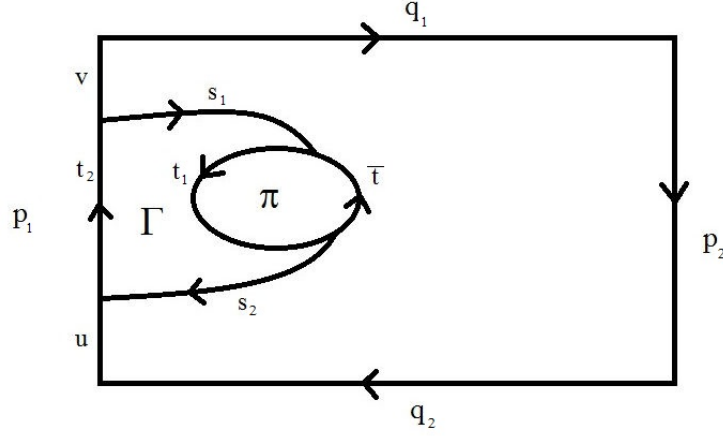


Figure IX.1: A single contiguity to a short side

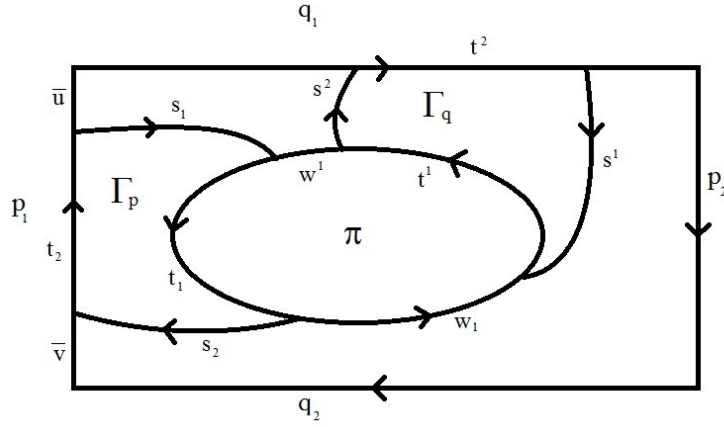


Figure IX.2: A contiguity to  $q_1$  and a contiguity to a short side

Note that  $2\zeta + \gamma < \bar{\beta}(2\beta - \gamma) - 2\zeta$  (LPP  $\gamma$ ), so  $|s^1 w_1^{-1} s_2| < |t_2|$ . Now, set  $\bar{p}_1 = \bar{v} s_2^{-1} w_1 (s^1)^{-1}$ . Now

$$|\bar{p}_1| + |p_2| < |\bar{v}| + |t_2| + |p_2| \leq |p_1| + |p_2| < \gamma |q_2|.$$

Thus, the result holds for subdiagram  $\bar{\Delta}$  with contour  $\bar{p}_1 v p_2 q_2$  and therefore for  $\Delta$ .

Case 2b: There are disjoint contiguity submaps  $\Gamma_p$  and  $\Gamma_q$  to  $p_1$  (or  $p_2$ ) and  $q_2$  respectively with  $(\pi, \Gamma_p, p_1) + (\pi, \Gamma_q, q_2) > \bar{\gamma}$ . Write  $\partial(\pi, \Gamma_p, p_1) = s_2 t_1 s_1 t_2$ ,  $\partial(\pi, \Gamma_q, q_2) = s^2 t^1 s^1 t^2$ ,  $\partial\pi = w_1 t_1 w^1 t^1$ ,  $p_1 = \bar{u} t_2 \bar{v}$ , and  $q_2 = v t^2 u$ .

By lemma II.21, we have  $(\pi, \Gamma_q, q_2) < \bar{\alpha}$ , so  $(\pi, \Gamma_p, p_1) > \bar{\gamma} - \bar{\alpha}$ . Therefore, by lemma II.19  $|t_2| > (\bar{\gamma} - \bar{\alpha} - 2\beta)|\partial\pi|$ . Set  $\bar{p}_1 = (s^1)^{-1} w_1 s_2^{-1} \bar{v}$ . Now,

$$|p_1| - |\bar{p}_1| = |\bar{u} t_2| - |s_2 w_1^{-1} s^1| > |\bar{u}| + (\bar{\gamma} - \bar{\alpha} - 2\beta - 2\zeta - \gamma)|\partial\pi| \quad (\text{IX.1})$$



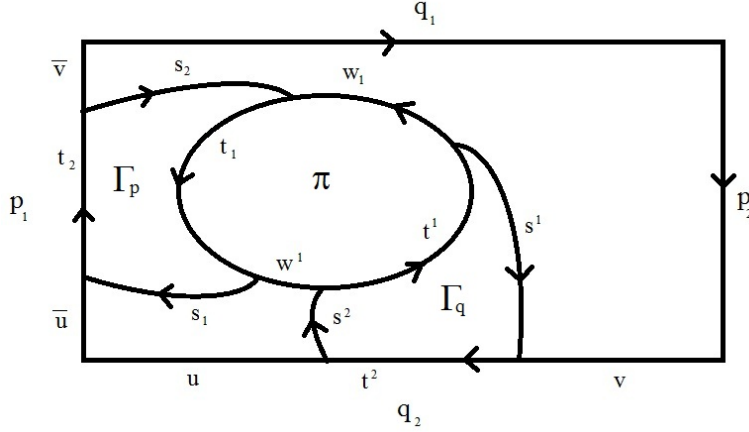


Figure IX.3: A contiguity to  $q_2$  and a contiguity to a short side

$$= |\bar{u}| + (\frac{1}{2} - 2\gamma - \alpha - 2\beta - 2\zeta) |\partial\pi|,$$

since  $|s^1|, |s_2| < \zeta|\partial\pi|$  by lemma II.18 and  $|w_1| < \gamma|\partial\pi|$ .

By theorem II.26,

$$|t^2 u| < \bar{\beta}^{-1} |\bar{u} s_1^{-1} w^1 t^1 s^1| < \bar{\beta}^{-1} (|\bar{u}| + (2\zeta + \gamma + \bar{\alpha}) |\partial \pi|) \quad (\text{IX.2})$$

since  $|s_1|, |s^1| < \zeta|\partial\pi|$  by lemma II.18,  $|w^1| < \gamma|\partial\pi|$  because  $\pi$  is a  $\gamma$ -cell, and  $|t^1| < \bar{\alpha}|\partial\pi|$  by lemma II.21.

Since  $\gamma\bar{\beta}^{-1} < 1$  and  $\gamma\bar{\beta}^{-1}(2\zeta + \gamma + \bar{\alpha}) < \frac{1}{2} - 2\gamma - \alpha - 2\beta - 2\zeta$  (LPP  $\gamma < \beta$ ), it follows from (IX.1) and (IX.2) that  $\gamma(|q_2| - |\nu|) = \gamma|t^2u| < |p_1| - |\bar{p}_1|$ . Hence,

$$|\bar{p}_1| + |p_2| = (|p_1| + |p_2|) - (|p_1| - |\bar{p}_1|) < \gamma(|q_2| - (|q_2| - |v|)) = \gamma|v|.$$

Now, the result holds for the submap  $\bar{\Delta}$  with contour  $\bar{p}_1 q_1 p_2 v$ , and therefore for  $\Delta$ .

Case 3: There are disjoint contiguity submaps  $\Gamma, \Gamma_1, \Gamma_2$  of  $\pi$  to  $p_1$  (or  $p_2$ ),  $q_1$ , and  $q_2$  respectively, with degrees summing to more than  $\bar{\gamma}$ . Write  $\partial(\pi, \Gamma_1, q_1) = p_1^1 q_1^1 p_2^1 q_2^1$ ,  $\partial(\pi, \Gamma_2, q_2) = p_1^2 q_1^2 p_2^2 q_2^2$ ,  $\partial(\pi, \Gamma, p_1) = s_1 t_1 s_2 t_2$ ,  $\partial \pi = w q_1^2 w' q_1^1 w'' t_1$ ,  $p_1 = \bar{p} t_2 \bar{p}$ ,  $q_2 = \bar{q}_2 q_2^2 \bar{q}_2$ , and  $q_1 = \bar{q}_1 q_2^1 \bar{q}_1$ .

Suppose  $(\pi, \Gamma, p_1) > \bar{\gamma} - \bar{\beta} = \beta - \gamma$  (otherwise, we have a  $\beta$ -cell as desired). By theorem II.26 and lemmas II.18 and II.21,

$$|q_2^2 \bar{q}_2| < \bar{\beta}^{-1} |\bar{p}(s_2)^{-1} w q_1^2 p_2^2| < \bar{\beta}^{-1} (|\bar{p}| + (2\zeta |\partial\pi| + |w| + |q_1^2|)) \\ < 2|\bar{p}| + (4\zeta + 2\gamma + 2\bar{\alpha})|\partial\pi| < 2|\bar{p}| + (1 + 3\alpha)|\partial\pi| \quad (\text{LPP } \beta).$$

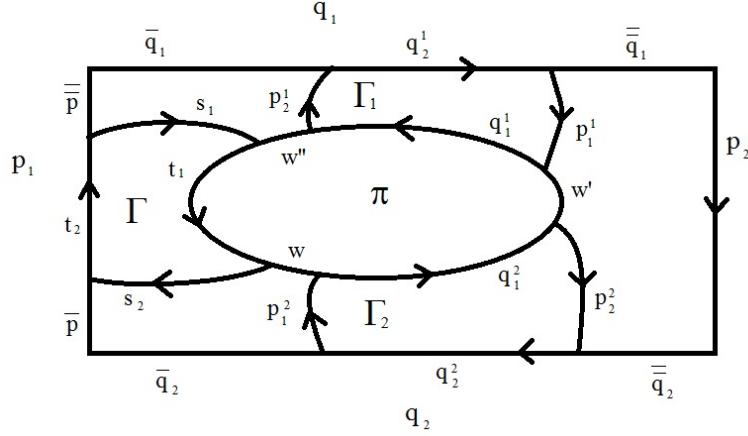


Figure IX.4: Three contiguities - only one of them to a short side

Set  $\bar{p}_1 = (p_2^2)^{-1}w'(p_1^1)^{-1}$ . Using the preceding inequality together with the facts  $|w'| < \gamma|\partial\pi|$  and  $|p_1^1|, |p_2^2| < \zeta|\partial\pi|$ , it follows that

$$\gamma|q_2^2\bar{q}_2| + |\bar{p}_1| < 2\gamma|\bar{p}| + ((1+3\alpha)\gamma + 2\zeta + \gamma)|\partial\pi| < |\bar{p}| + (2\zeta + 3\gamma)|\partial\pi| < |\bar{p}| + \frac{\beta}{2}|\partial\pi|.$$

By lemma II.19,  $|t_2| > (\bar{\beta}(\beta - \gamma) - 2\zeta)|\partial\pi| > \frac{\beta}{2}|\partial\pi|$  so that  $\gamma|q_2^2\bar{q}_2| + |\bar{p}_1| < |t_2| + |\bar{p}| \leq |p_1|$ . Now  $\gamma|\bar{q}_2| = \gamma(|q_2| - |q_2^2\bar{q}_2|) > |p_1| + |p_2| - (|p_1| - |\bar{p}_1|) = |\bar{p}_1| + |p_2|$ . Therefore, we can apply the induction hypothesis to the diagram  $\bar{\Delta}$  with contour  $\bar{p}_1\bar{q}_1p_2\bar{q}_2$ .

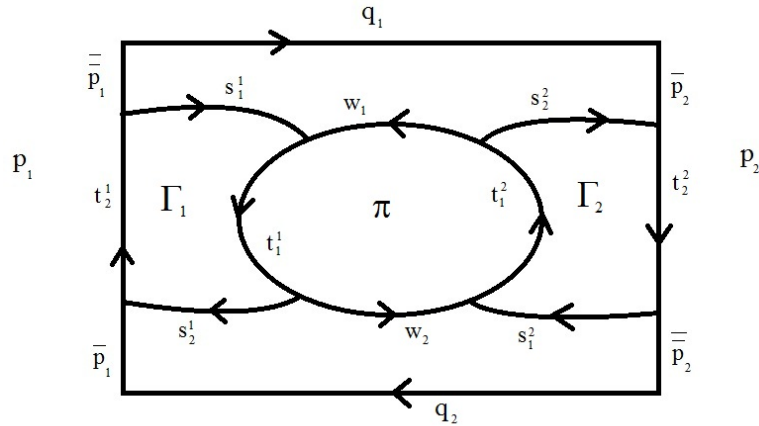


Figure IX.5: Contiguities to both short sides

Case 4: There are contiguity subdiagrams  $\Gamma_1$  and  $\Gamma_2$  of  $\pi$  to  $p_1$  and  $p_2$  respectively with  $(\pi, \Gamma_1, p_1) + (\pi, \Gamma_2, p_2) > \bar{\gamma} - \bar{\beta} = \beta - \gamma$ . Set  $\partial(\pi, \Gamma_1, p_1) = s_1^1 t_1^1 s_2^1 t_2^1$ ,  $\partial(\pi, \Gamma_2, p_2) = s_1^2 t_1^2 s_2^2 t_2^2$ ,  $\partial\pi = w_1 t_1^1 w_2 t_1^2$ , and  $p_i = \bar{p}_i t_2^i \bar{p}_i$ .

By lemma II.18 and theorem II.26,

$$|t_2^1| + |t_2^2| > \bar{\beta}(|t_1^1| + |t_1^2|) - 4\zeta|\partial\pi| > (\bar{\beta}(\beta - \gamma) - 4\zeta)|\partial\pi|.$$

Now,

$$\begin{aligned} |p_1| + |p_2| &\geq |t_2^1| + |t_2^2| + |\bar{p}_1| + |\bar{p}_2| > (\bar{\beta}(\beta - \gamma) - 4\zeta)|\partial\pi| + |\bar{p}_1| + |\bar{p}_2| \\ &> 2\beta|\partial\pi| + |\bar{p}_1| + |\bar{p}_2|. \end{aligned}$$

Applying theorem II.26 to the map with boundary  $\bar{p}_1(s_2^1)^{-1}w_2(s_1^2)^{-1}\bar{p}_2q_2$  we obtain

$$\begin{aligned} \gamma|q_2| &\leq \gamma\bar{\beta}^{-1}|\bar{p}_1(s_2^1)^{-1}w_2(s_1^2)^{-1}\bar{p}_2| \\ &< \gamma\bar{\beta}^{-1}(|\bar{p}_1| + |\bar{p}_2| + (2\zeta + 1 + \gamma - \beta)|\partial\pi|) < |\bar{p}_1| + |\bar{p}_2| + 2\beta|\partial\pi|. \end{aligned}$$

But these estimates contradict  $\gamma|q_2| > |p_1| + |p_2|$ .

Now, we claim that these are all the possible cases. Indeed, if  $\Delta$  has positive rank, then corollary II.23 tells us there are distinct contiguity submaps  $\Gamma^1, \dots, \Gamma^4$  (some may be absent) to distinct sections, with degrees summing to more than  $\bar{\gamma}$ . Case 4 rules out the possibility that there are contiguities to  $p_1$  and  $p_2$  with degrees large enough to prevent the existence of a  $\beta$ -cell. In particular, if all four of the contiguity submaps are present, we have a  $\beta$ -cell. Case 1 deals with the possibility that only one of the contiguity submaps mentioned above is present. Case 2 and case 4 cover the possibility that exactly two of the contiguity submaps are present, while cases 3 and 4 cover the possibility that exactly 3 are present. These are all the possibilities.  $\square$

**Lemma IX.3.** *Let  $\Delta$  be an A-map with contour  $p_1q_1p_2q_2$  where  $|p_1| + |p_2| < \gamma|q_2|$ ,  $q_2$  is a smooth section of rank  $k$ , and  $q_1$  is  $(1 - 2\beta)$ -reduced in  $\Delta$ . Then*

- 1) *there exist vertices  $o_1$  and  $o_2$  on  $q_1$  and  $q_2$  and a path  $x$  joining them, with  $|x| < 2k$ ;*
- 2) *we can choose  $o_1$  and  $o_2$  in such a way that the initial segment of  $q_2^{-1}$  (or of  $q_2$ ) up to  $o_2$  is less than  $\gamma^{-1}(|p_1| + 2k)$  (or than  $\gamma^{-1}(|p_2| + 2k)$ ).*

*Proof.* 1) If there is a 0-bond between  $q_1$  and  $q_2$ , we can simply take  $x$  to be a side arc of the bond, in which case  $|x| = 0$ . Otherwise, by lemma IX.2,  $\Delta$  has a  $\beta$ -cell  $\pi$  with contiguities  $\Gamma_1$  and  $\Gamma_2$  to  $q_1$  and  $q_2$ , respectively, satisfying  $(\pi, \Gamma_1, q_1) + (\pi, \Gamma_2, q_2) > \bar{\beta}$ . Write  $\partial(\pi, \Gamma_1, q_1) = s_2^1 t_1^1 s_1^1 t_2^1$ ,  $\partial(\pi, \Gamma_2, q_2) = s_2^2 t_1^2 s_1^2 t_2^2$ , and  $\partial\pi = t_1^1 u_2 t_1^2 u_1$ .

Since  $(\pi, \Gamma_1, q_1) < 1 - 2\beta < \bar{\beta}$ , and  $(\pi, \Gamma_2, q_2) < \bar{\alpha} < \bar{\beta}$ , it follows that both  $\Gamma_1$  and  $\Gamma_2$  exist. We have  $|s_1^1|, |s_2^2| < \zeta|\partial\pi|$  by lemma II.18 and  $|u_1|, |u_2| < \beta|\partial\pi|$ . Taking  $x = s_2^2 u_2^{-1} s_1^1$  (or  $x = s_2^1 u_1^{-1} s_1^2$ ), we have  $|x| < (2\zeta + \beta)|\partial\pi|$ .

Now, since  $(\pi, \Gamma_1, q_1) < 1 - 2\beta$ , we have  $(\pi, \Gamma_2, q_2) > \beta > \varepsilon$ . By lemma II.19,  $|t_2^2| > (\bar{\beta}\beta - 2\zeta)|\partial\pi|$ . By definition of a smooth section,  $|t_2^2| < (1 + \gamma)k$ . So,  $|x| < (2\zeta + \beta)(\bar{\beta}\beta - 2\zeta)^{-1}(1 + \gamma)k < 2k$ .

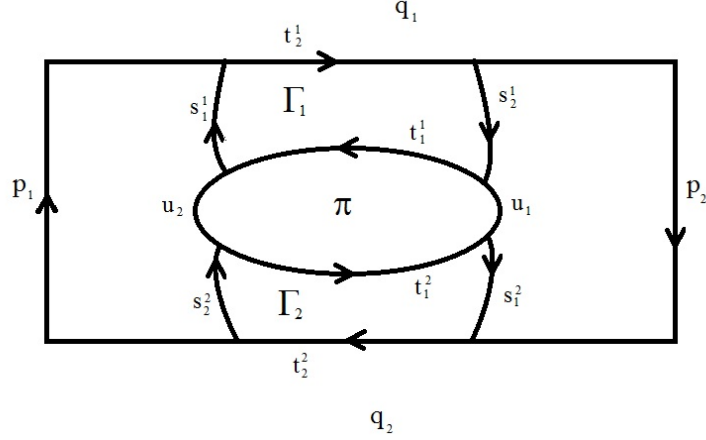


Figure IX.6: A  $\beta$ -cell

2) Write  $q_1 = u_1 v_1$  and  $q_2^{-1} = u_2^{-1} v_2^{-1}$ , where  $o_1$  and  $o_2$  are the terminal points of  $u_1$  and  $u_2^{-1}$ , such that  $o_1, o_2$  are as in part 1, with  $|u_1|$  minimal. Then there are no  $\beta$ -cells or 0-bonds between  $u_1$  and  $u_2$  in the submap with contour  $p_1 u_1 x u_2$ . Therefore, by lemma IX.2, we have  $|p_1| + 2k > |p_1| + |x| \geq \gamma |u_2|$ .  $\square$

**Lemma IX.4.** *Let  $\Delta$  be an A-map with contour  $p_1 q_1 p_2 q_2$  where  $|p_1|, |p_2| < \zeta nk$ ,  $q_2$  is a smooth section of rank  $k$ , and  $q_1$  is  $(1 - 2\beta)$ -reduced in  $\Delta$ .*

i) *Then, we can divide  $\Delta$  into submaps  $\Delta_i$  for  $i = 1, \dots, l$  with  $\partial \Delta_i = p'_i q_1^i (p'_{i+1})^{-1} q_2^i$ , where  $p'_1 = p_1$ ,  $p'_{l+1} = (p_2)^{-1}$ ,  $|p'_i| < \zeta nk$ , and  $|q_2^i| < 2\zeta nk \gamma^{-1}$ .*

*Additionally, if  $|q_2| \geq \beta nk$ , we have*

ii)  $\gamma^{-1} |(p'_{i+1})^{-1} q_2^i p'_i| < |q_1| - |q_1^i|$ . *In particular,  $|\partial \Delta_i| < |q_1|$  for each  $i$ . Also,  $\beta nk < 2(|q_1| - |q_1^i|)$ .*

*Proof.* i) By repeatedly applying the preceding lemma, we divide  $\Delta$  into submaps  $\Delta_i$  with  $\partial \Delta_i = p'_i q_1^i (p'_{i+1})^{-1} q_2^i$ , where  $p'_1 = p_1$ ,  $p'_{l+1} = (p_2)^{-1}$ ,  $|p'_i| < \zeta nk$  (since  $2k < \zeta nk$ ), and  $|q_2^i| < 2\zeta nk \gamma^{-1}$ .

ii) If  $|q_2| \geq \beta nk$ , then

$$|q_2| - |q_2^i| \geq (\beta - 2\zeta \gamma^{-1}) nk. \quad (\text{IX.3})$$

The subpaths  $q_2^{i-1} \dots q_2^1$  and  $q_2^l \dots q_2^{i+1}$  of the smooth section  $q_2$  of rank  $k$  are themselves smooth of rank  $k$ , by lemma II.17. Therefore, we can apply theorem II.26 to the maps with contours  $p_1 q_1^1 q_1^2 \dots q_1^{i-1} (p'_i)^{-1} q_2^{i-1} \dots q_2^1$  and  $p'_{i+1} q_1^{i+1} \dots q_1^l p_2 q_2^l \dots q_2^{i+1}$ , and we obtain

$$\bar{\beta} |q_2^{i-1} \dots q_2^1| \leq |p_1| + |p'_i| + |q_1^1 q_1^2 \dots q_1^{i-1}| < 2\zeta nk + |q_1^1 q_1^2 \dots q_1^{i-1}|$$

and similarly

$$\bar{\beta} |q_2^l \dots q_2^{i+1}| < 2\zeta nk + |q_1^{i+1} \dots q_1^l|.$$

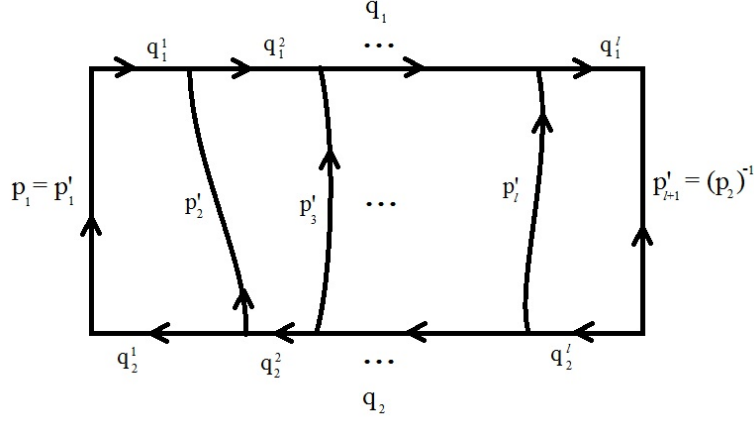


Figure IX.7: An A-map cut into thin pieces

Adding these inequalities and using the fact that  $|q_2^{i-1} \cdots q_2^1| + |q_2^i \cdots q_2^{i+1}| = |q_2| - |q_2^i|$  and  $|q_1^1 q_1^2 \cdots q_1^{i-1}| + |q_1^{i+1} q_1^{i+2} \cdots q_1^i| = |q_1| - |q_1^i|$ , we obtain

$$\bar{\beta}(|q_2| - |q_2^i|) < 4\zeta nk + |q_1| - |q_1^i|.$$

Combining this inequality with (IX.3) and LPP ( $\zeta < \gamma$ ,  $\beta < \alpha$ ), we obtain

$$\frac{\beta nk}{2} < (\bar{\beta}(\beta - 2\zeta\gamma^{-1}) - 4\zeta)nk < |q_1| - |q_1^i|.$$

By the estimates in part 1 and LPP ( $\zeta < \gamma$ ),

$$\gamma^{-1}(|q_2^i| + |p_i'| + |p_{i+1}'|) < 2\zeta nk\gamma^{-1}(\gamma^{-1} + 1) < \frac{\beta nk}{2}.$$

Combining the last two inequalities yields the desired result,

$$|(p_{i+1}')^{-1} q_2^i p_i'| < \gamma(|q_1| - |q_1^i|)$$

and it follows immediately that

$$|\partial\Delta_i| = |(p_{i+1}')^{-1} q_2^i p_i'| + |q_1^i| < \gamma(|q_1| - |q_1^i|) + |q_1^i| < (|q_1| - |q_1^i|) + |q_1^i| = |q_1|.$$

□

**Lemma IX.5.** *Suppose that  $\Delta$  is a circular A-map. Suppose further that 0-cells are the only subdiagrams with contour labels consisting only of 0-cells. If  $o$  is a vertex of  $\Delta$  there is a path from  $o$  to  $\partial\Delta$  with length  $\leq \gamma|\partial\Delta|$ .*

*Proof.* We proceed by induction on the number of  $R$ -cells. Suppose  $r(\Delta) = 0$ . First, there is a path  $t$  of

length zero from  $o$  to an endpoint  $o'$  of some  $S$ -edge  $e$  (or to the contour, in which case we're done). Now,  $e$  is joined by a maximal (i.e. not a proper subset of any other) 0-bond to some edge  $f$ . We claim that  $f$  must lie on the contour of  $\Delta$ . Indeed, if  $f$  were not on  $\partial\Delta$ , then either the side arcs of the bond would form an annulus or the bond could be extended past  $f$ ; The latter case is clearly a contradiction. In the former case, one of the side arcs of the 0-bond is not the contour of a 0-cell; contradiction. Therefore,  $f$  is on  $\partial\Delta$ , as claimed.

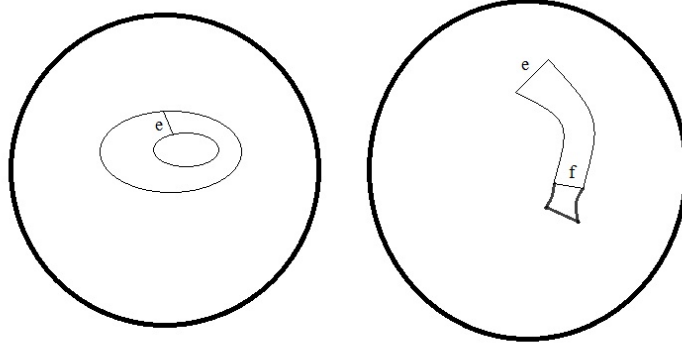


Figure IX.8: Some impossible maximal zero-bonds

Now we can choose a path  $t'$  from  $o'$  to a vertex of  $\partial\Delta$  where  $t'$  is a side arc of a 0-bond. So,  $tt'$  is a path from  $o'$  to  $\partial\Delta$  with length  $0 < \gamma|\partial\Delta|$ .

Now, suppose  $r(\Delta) > 0$ . Write  $\partial\Delta = q$ . By corollary II.23,  $\Delta$  has a  $\gamma$ -cell  $\Pi$ . Hence, there is a contiguity submap of  $\Pi$  to  $q$  with degree at least  $\bar{\gamma}$ . Therefore, there must be a contiguity submap  $\Gamma$  of  $\Pi$  to  $q$  with  $(\Pi, \Gamma, q) > 1 - 2\beta$  and  $q_1 = \Gamma \wedge q$  is  $(1 - 2\beta)$ -reduced in  $\Gamma$ . Indeed, we can see this as follows: begin with a  $\gamma$ -cell  $\Pi^1$  contiguous to  $q$  across a contiguity submap  $\Gamma^1$ . If  $q^1 = \Gamma^1 \wedge q$  is  $(1 - 2\beta)$ -reduced in  $\Gamma^1$ , we have our desired cell. Otherwise, there is an  $R$ -cell  $\Pi^2$  in  $\Gamma^1$  which is contiguous to  $q^1$  (and therefore to  $q$ ) across a contiguity submap  $\Gamma^2$  with  $(\Pi^2, \Gamma^2, q^1) > 1 - 2\beta$ . Since there are only finitely many  $R$ -cells in  $\Delta$ , we must eventually obtain our desired  $\Gamma$ . Write  $\partial(\Pi, \Gamma, q) = p_2 q_2 p_1 q_1$ , let  $q'_2$  be the complement of  $q_2$  in  $\partial\Pi$ . Let  $q'_1$  be the complement of  $q_1$  in  $\partial\Delta$ .

Case 1 -  $o \notin \Gamma$ :

By lemmas II.18 and II.19, we have

$$|p_1|, |p_2| < \zeta|\partial\Pi| \quad \text{and} \quad |q_1| > (1 - 4\beta)|\partial\Pi|. \quad (\text{IX.4})$$

Additionally, since  $(\Pi, \Gamma, q) > 1 - 2\beta$ ,  $|q'_2| < 2\beta|\partial\Pi|$ . Now, the circular submap with contour  $q'_1 p_1^{-1} q'_2 p_2^{-1}$  has fewer  $R$ -cells than  $\Delta$ , so we can apply the induction hypothesis and (IX.4) to find that there is a path  $t_0$

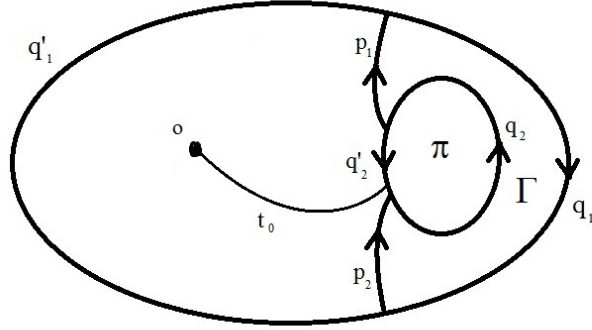


Figure IX.9: The case where  $o$  is outside the contiguity

from  $o$  to a point  $o'$  on  $q'_1 p_1^{-1} q'_2 p_2^{-1}$  in  $\Delta$  with

$$|t_0| \leq \gamma |q'_1 p_1^{-1} q'_2 p_2^{-1}| = \gamma(|\partial\Delta| - |q_1| + |p_1| + |q'_2| + |p_2|) \quad (\text{IX.5})$$

$$< \gamma(|\partial\Delta| + (2\zeta + 6\beta - 1)|\partial\Pi|).$$

Using our estimates for  $|p_1|$ ,  $|p_2|$ , and  $|q'_2|$  again, we can extend (if necessary - i.e. if  $o'$  is on  $p_1^{-1} q'_2 p_2^{-1}$ ) the path  $t_0$  to  $\partial\Delta$  by adding no more than  $\frac{1}{2}|q'_2| + |p_i| < (\beta + \zeta)|\partial\Pi|$ . Therefore, by (IX.5) and the last estimate, there is a path from  $o$  to  $\partial\Delta$  whose length is less than

$$\gamma(|\partial\Delta| + (3\zeta + 7\beta - 1)|\partial\Pi|) < \gamma|\partial\Delta| \quad (\text{LPP } \beta < 1).$$

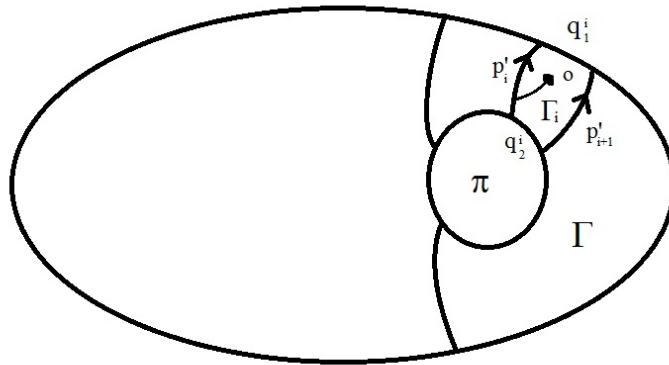


Figure IX.10: The case where  $o$  is in the contiguity

Case 2 -  $o \in \Gamma$ :

By lemma II.17,  $q_2$  is a smooth section of rank  $r(\Pi) = k$  and  $\Gamma$  is an  $A$ -map. Then lemma II.18 tells us that  $|p_1|, |p_2| < \zeta nk$ . Since  $q_1$  is  $(1 - 2\beta)$ -reduced in  $\Gamma$ , we can apply lemma IX.4, dividing  $\Gamma$  into submaps  $\Gamma_i$  for  $i = 1, \dots, l$  with  $\partial\Gamma_i = p'_i q_1^i (p'_{i+1})^{-1} q_2^i$  where  $p'_1 = p_1$ ,  $p'_{l+1} = (p_2)^{-1}$ ,  $|p'_i| < \zeta nk$ , and  $|q_2^i| < 2\zeta nk \gamma^{-1}$ . Additionally, since  $|q_2| > (1 - 2\beta)|\partial\Pi| \geq (1 - 2\beta)nk > \tilde{\alpha}nk > \beta nk$ , we may assume  $\gamma^{-1}|(p_{i+1})^{-1} q_2^i p'_i| < |q_1| - |q_1^i|$ . Now,  $o$  is in some  $\Gamma_i$ , and  $\Gamma_i$  has fewer  $R$ -cells than  $\Delta$ . Therefore, by the induction hypothesis, there is a path  $t_0$  from  $o$  to a vertex  $o'$  on  $\partial\Gamma_i$  with length

$$\leq \gamma|\partial\Gamma_i| = \gamma(|q_1^i| + |(p_{i+1})^{-1} q_2^i p'_i|) < \gamma(|q_1^i| + \gamma(|q_1| - |q_1^i|)) \quad (\text{IX.6})$$

Now, if necessary, we can extend the path  $t_0$  to  $q_1^i$  without increasing its length by more than  $\frac{1}{2}|(p_{i+1})^{-1} q_2^i p'_i| < \frac{1}{2}\gamma(|q_1| - |q_1^i|)$ . Combining this estimate with (IX.6), there is a path from  $o$  to  $\partial\Delta$  whose length does not exceed

$$\gamma\left(\frac{1}{2} + \gamma\right)(|q_1| - |q_1^i|) + |q_1^i| < \gamma|q_1| \leq |\partial\Delta|.$$

□



## CHAPTER X

### ISODIAMETRIC AND SPACE FUNCTIONS

**Definition X.1.** Let  $G$  be a group with presentation  $\langle S|R \rangle$ . We define an  $R$ -transformation to be any composition of operations of the following types:

- 1) the insertion or deletion of a pair of adjacent mutually inverse letters, or
- 2) replacing a subword  $u$  with a word  $v$ , where  $uv^{-1}$  is a cyclic shift of  $r^{\pm 1}$  where  $r \in R$ .

**Definition X.2.** A function  $f$  is a filling length function for the presentation  $\langle S|R \rangle$  of a group  $G$  if, for each word  $w = 1$  in  $G$ , there is a sequence of  $R$ -transformations

$$w \equiv w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_m \equiv 1$$

with  $|w_i| \leq f(|w|)$  for all  $i$ .

**Lemma X.3.** Let  $\Delta$  be a diagram over  $G = \langle S|R \rangle$  whose underlying map is an  $A$ -map with contour  $p_1 q_1 p_2 q_2$  where  $q_2$  is a smooth section of rank  $k$ ,  $q_1$  is  $(1 - 2\beta)$ -reduced in  $\Delta$ ,  $\max(|p_1|, |p_2|) < \zeta nk$ , and  $|q_2| \geq \beta nk$ . Suppose further that, if  $w, v$  are words over  $S$  with  $|w| + |v| < |q_1|$  and  $w = v$  in  $G$ , then there is a chain of  $R$ -transformations  $w \rightarrow v$  in which no word has length exceeding  $(1 + 2\beta)(|w| + |v|)$ . Then there is a chain of  $R$ -transformations  $Lab(q_1) \rightarrow Lab(p_1^{-1} q_2^{-1} p_2^{-1})$  in which no word has length exceeding  $(1 + 2\beta)|q_1| - 10\gamma nk$ .

*Proof.* Throughout this proof, write  $D = 1 + 2\beta$ . By lemma IX.4, we can divide  $\Delta$  into subdiagrams  $\Delta_1, \dots, \Delta_l$  with the properties given in the conclusion of lemma IX.4. In order to simplify our notation, let us write  $q_1^{(u,v)} = q_1^u q_1^{u+1} \cdots q_1^v$  and  $q_2^{(u,v)} = q_2^v q_2^{v-1} \cdots q_2^u$  for  $u \leq v$ . Also, let  $q_2^{(1,0)}$  denote constant path consisting only of the point  $(q_2)_+$ . Now, by lemma IX.4,  $|\partial \Delta_i| < |q_1|$ . In other words,  $|p'_i q_1^i| + |(q_2^i)^{-1} p_{i+1}| < |q_1|$ . Thus, there is a chain of  $R$ -transformations  $Lab(p'_i q_1^i) \rightarrow Lab((q_2^i)^{-1} p'_{i+1})$  in which no word has length exceeding  $D|\partial \Delta_i|$ . Therefore, there is a chain of  $R$ -transformations

$$Lab(p_1^{-1} (q_2^{(1,i-1)})^{-1} p'_i q_1^{(i,l)}) \rightarrow Lab(p_1^{-1} (q_2^{(1,i)})^{-1} p'_{i+1} q_1^{(i+1,l)}) \quad (\text{X.1})$$

in which no word has length exceeding

$$\begin{aligned} |p_1| + |q_2^{(1,i-1)}| + |q_1^{(i+1,l)}| + D|\partial \Delta_i| &= |p_1| + |q_2^{(1,i-1)}| + |q_1^{(i+1,l)}| + D|q_1^i| + D|(p'_{i+1})^{-1} q_2^i p'_i| \\ &< \zeta nk + |q_2^{(1,i-1)}| + |q_1^{(i+1,l)}| + D|q_1^i| + D\gamma(|q_1| - |q_1^i|), \end{aligned} \quad (\text{X.2})$$

where the inequality  $|(p'_{i+1})^{-1} q_2^i p'_i| < \gamma(|q_1| - |q_1^i|)$  is from lemma IX.4.

By lemma II.17,  $q_2^{(1,i-1)}$  is a smooth section of the  $A$ -map with contour  $p_1 q_1^{(1,i-1)} (p'_i)^{-1} q_2^{(1,i-1)}$ .

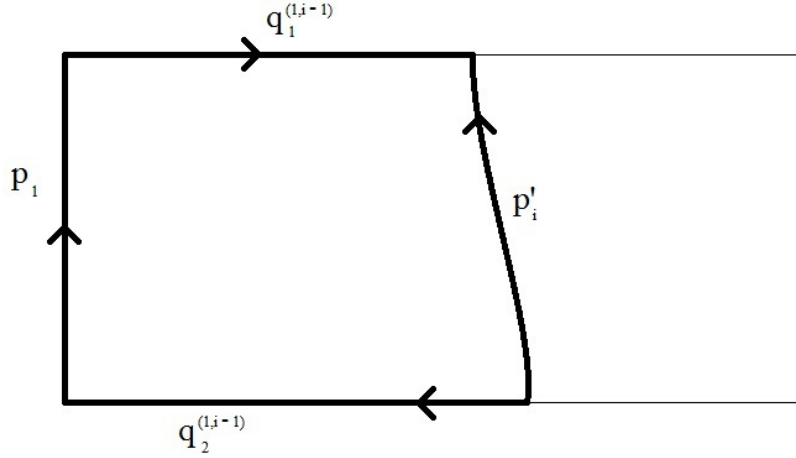


Figure X.1: An intermediate stage of pushing across the narrow diagram

Therefore, by theorem II.26,

$$\begin{aligned}
 |q_2^{(1,i-1)}| &\leq \bar{\beta}^{-1}(|p_1| + |p'_i| + |q_1^{(1,i-1)}|) < 2\bar{\beta}^{-1}\zeta nk + \bar{\beta}^{-1}|q_1^{(1,i-1)}| \\
 &< 3\zeta\beta nk + \bar{\beta}^{-1}|q_1^{(1,i-1)}|.
 \end{aligned}$$

Combining this inequality with (X.2) and using lemma IX.4, we see that no word in (X.1) has length greater than

$$\begin{aligned}
 &\bar{\beta}^{-1}|q_1^{(1,i-1)}| + |q_1^{(i+1,l)}| + D|q_1^i| + D\gamma(|q_1| - |q_1^i|) + (3 + \beta^{-1})\zeta\beta nk \\
 &< \bar{\beta}^{-1}(|q_1^{(1,i-1)}| + |q_1^{(i+1,l)}|) + D|q_1^i| + (D\gamma + (6 + 2\beta^{-1})\zeta)(|q_1| - |q_1^i|) \\
 &< (D\gamma + \bar{\beta}^{-1} + 3\beta^{-1}\zeta)(|q_1| - |q_1^i|) + D|q_1^i| \\
 &D|q_1| - 20\gamma\beta^{-1}(|q_1| - |q_1^i|) < D|q_1| - 10\gamma nk.
 \end{aligned}$$

Now, note that there is a chain of  $R$ -transformations

$Lab(q_1) \rightarrow Lab(p_1^{-1}(q_2^{(1,0)})^{-1}p'_1q_1^{(1,l)})$  consisting only of insertions of letters. Therefore, following this chain with the chains in (X.1) for  $i = 1, \dots, l$ , we obtain a chain  $Lab(q_1) \rightarrow Lab(p_1^{-1}q_2^{-1}p_2^{-1})$  in which no word has length exceeding  $D|q_1| - 10\gamma nk$ .  $\square$

**Theorem X.4.** Suppose that  $w, v$  are words over  $S$  with  $w = v$  in  $B_C(S, n)$ . Then there is a chain  $w \rightarrow v$  of  $R$ -transformations in which no word has length greater than  $(1 + 2\beta)\max(|w|, |v|)$ . Furthermore, we may choose the chain in such a way that every word consists only of  $S$ -letters that appear in  $w$  or  $v$ .

*Proof.* We will proceed by induction on  $|w| + |v|$ . Since  $wv^{-1} = 1$  in  $G_C(\infty)$ , there is a reduced circular diagram  $\Delta$  with contour  $pq$  where  $Lab(p) \equiv w$  and  $Lab(q) \equiv v^{-1}$ . According to lemma IV.2, every  $S$ -letter

that labels an edge of an  $R$ -cell appears in  $w$  or  $v$ . If any  $S$ -letter labels an edge in the diagram but does not appear in  $w$ ,  $v$ , or the label of an  $R$ -cell, then this  $S$ -letter can be removed from  $\Delta$ . Therefore, we may assume that every  $S$ -letter that labels an edge in  $\Delta$  appears in  $w$  or  $v$ . If  $\Delta$  contains an  $R$ -cell  $\Pi$ , then  $\bar{\beta}n \leq \bar{\beta}nr(\Pi) \leq \bar{\beta}|\partial\Pi| \leq |\partial\Delta|$  by corollary II.27. In particular, if  $|\partial\Delta| < \bar{\beta}n$ , then  $r(\Delta) = 0$ .

If  $r(\Delta) = 0$ , then  $w$  and  $v$  are freely equal. In particular,  $w$  and  $v$  have the same reduced form  $u$ . Hence, there is a chain of  $R$ -transformations

$$w \rightarrow u \rightarrow v$$

in which we pass from  $w$  to  $u$  using only deletions and then from  $u$  to  $v$  using only insertions of adjacent mutually inverse letters. In particular, no word in this chain has length exceeding  $\max(|w|, |v|)$ . This will serve as the base of our induction. For the remainder of this proof, take  $D = 1 + 2\beta$ .

On the other hand, if  $r(\Delta) > 0$ , then  $\Delta$  contains a  $\gamma$ -cell by corollary II.23. We will now proceed in two cases.

Case 1 - There is an  $R$ -cell  $\Pi_0$  contiguous across  $\Gamma_0$  to  $p$  (or  $q$ ) with contiguity degree at least  $1 - 2\beta$ . If  $p^0 = \Gamma_0 \wedge p$  is not  $(1 - 2\beta)$ -reduced in  $\Gamma_0$ , there is an  $R$ -cell  $\Pi_1$  in  $\Gamma_0$  with  $\Pi_1$  contiguous across  $\Gamma_1 \subset \Gamma_0$  to  $p$  with degree at least  $1 - 2\beta$ . We continue this procedure (it must end since  $\Gamma_i(2) < \Gamma_{i-1}(2)$ ) until we obtain an  $R$ -cell  $\Pi$  contiguous across  $\Gamma$  to  $p$ , with  $(\Pi, \Gamma, p) \geq 1 - 2\beta$  and  $q_1 = \Gamma \wedge p$  is  $(1 - 2\beta)$ -reduced in  $\Gamma$ . Write  $\partial(\Pi, \Gamma, p) = p_2 q_2 p_1 q_1$ . Write  $p = p' q_1 p''$  and  $\partial\Pi = q_2 q'_2$ .

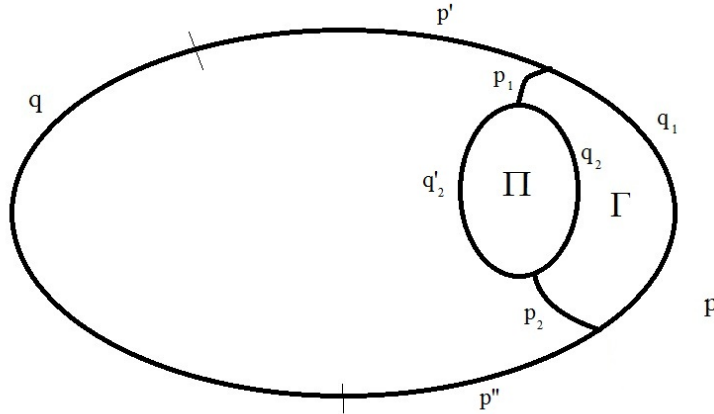


Figure X.2: The case with a contiguity to the contour with degree at least  $1 - 2\beta$

By lemma II.17,  $q_2$  is a smooth section of rank  $r(\Pi) = k$  in  $\Gamma$ . By lemma II.18,  $|p_1|, |p_2| < \zeta nk$ . Since  $|q_1| \leq |\partial\Delta| = |w| + |v|$ , our induction hypothesis permits us to apply by lemma X.3, and there is a chain of  $R$ -transformations

$$Lab(q_1) \rightarrow Lab(p_1^{-1} q_2^{-1} p_2^{-1})$$

in which no word has length exceeding  $D|q_1|$ . Replacing  $Lab(q_2^{-1})$  by  $Lab(q'_2)$  is an  $R$ -transformation that

reduces length since  $|q_2| \geq (1 - 2\beta)|\partial\Pi| > 2\beta|\partial\Pi| > |q'_2|$ . Therefore, there is a chain of  $R$ -transformations

$$Lab(p) \equiv Lab(p'q_1p'') \rightarrow Lab(p'p_1^{-1}q'_2p_2^{-1}p'')$$

in which no word has length greater than  $|p'| + D|q_1| + |p''| \leq D|p|$ . By lemma II.20,  $|p_1^{-1}q'_2p_2^{-1}| < |q_1|$ , and therefore  $|p'p_1^{-1}q'_2p_2^{-1}p''| < |p'q_1p''| = |\partial\Delta|$ . Hence, the induction hypothesis guarantees that there is a chain of  $R$ -transformations  $Lab(p'p_1^{-1}q'_2p_2^{-1}p'') \rightarrow Lab(q)$  with no word's length exceeding  $D\max(|p'p_1^{-1}q'_2p_2^{-1}p''|, |q|) \leq D\max(|p|, |q|) = D\max(|w|, |v|)$ .

Case 2 - There is no  $R$ -cell contiguous to  $p$  or  $q$  with degree at least  $1 - 2\beta$ . In this case, corollary II.24 says there is a  $\gamma$ -cell  $\Pi$  with contiguities  $\Gamma_p, \Gamma_q$  to  $p, q$  respectively, such that  $(\Pi, \Gamma_p, p) + (\Pi, \Gamma_q, q) > \bar{\gamma}$ . Write  $\partial(\Pi, \Gamma_p, p) = p_2^1 q_2^1 p_1^1 q_1^1$  and  $\partial(\Pi, \Gamma_q, q) = p_2^2 q_2^2 p_1^2 q_1^2$ . Write  $\partial\Pi = u_1 q_2^1 u_2 q_2^2$ ,  $p = p' q_1^1 p''$ , and  $q = q' q_1^2 q''$ .

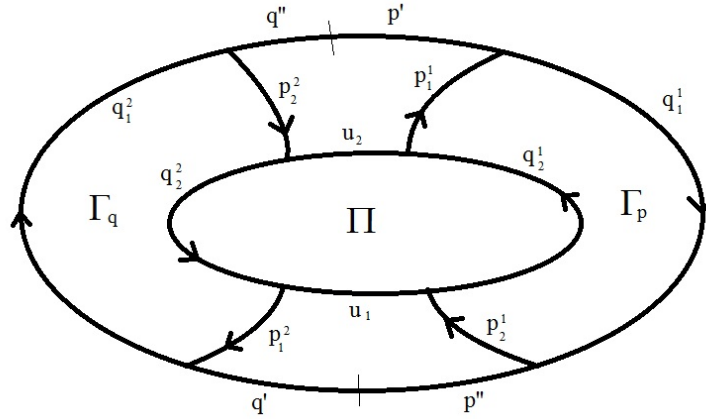


Figure X.3: The case where there are two contiguities to sections of the contour

Since  $(\Pi, \Gamma_p, p), (\Pi, \Gamma_q, q) < 1 - 2\beta$ , it follows that  $(\Pi, \Gamma_p, p), (\Pi, \Gamma_q, q) > \bar{\gamma} - (1 - 2\beta) = 2\beta - \gamma > \beta$ . By lemma II.17,  $q_2^1$  and  $q_2^2$  are smooth sections of rank  $r(\Pi) = k$  in  $\Gamma_p$  and  $\Gamma_q$ , respectively. Without loss of generality, we may assume that  $|q_1^1| \geq |q_1^2|$  and that  $|q''| - |p'| \leq |q'| - |p''|$ . By lemma II.18,  $|p_i^j| < \zeta nk$ . Since  $|q_1^1|, |q_1^2| \leq |\partial\Delta| = |w| + |v|$ , our induction hypothesis allows us to apply lemma X.3 to  $\Gamma_p$  and  $\Gamma_q$ . First, there is a chain of  $R$ -transformations

$$Lab(q_1^1) \rightarrow Lab((p_1^1)^{-1}(q_2^1)^{-1}(p_2^1)^{-1}) \quad (X.3)$$

in which no word has length exceeding  $D|q_1^1| - 10\gamma nk$ . Likewise, there is a chain of  $R$ -transformations

$$Lab(q_1^2) \rightarrow Lab((p_1^2)^{-1}(q_2^2)^{-1}(p_2^2)^{-1})$$

in which no word has length greater than  $D|q_1^2| - 10\gamma nk \leq D|q_1^1| - 10\gamma nk$ . Set  $t_1 = (p_1^1)^{-1}u_2(p_2^2)^{-1}$  and  $t_2 = (p_1^2)^{-1}u_1(p_2^1)^{-1}$ . Now, replacing  $Lab((q_2^1)^{-1})$  by  $Lab(u_2 q_2^2 u_1)$  is a single  $R$ -transformation, and neither

word has length greater than

$$D|q_1^1| - 10\gamma nk + |u_1| + |u_2| < D|q_1^1| - 9\gamma nk.$$

Combining this with (X.3), we have a chain

$$Lab(q_1^1) \rightarrow Lab(t_1(q_1^2)^{-1}t_2)$$

in which no word has length greater than  $D|q_1^1| - 9\gamma nk$ . Now, this gives us a chain of  $R$ -transformations

$$Lab(p) \equiv Lab(p'q_1^1p'') \rightarrow Lab(p't_1(q_1^2)^{-1}t_2p'') \quad (\text{X.4})$$

in which no word has length exceeding

$$|p'| + D|q_1^1| + |p''| - 9\gamma nk \leq D|p|.$$

Now, by lemma II.19,  $|q_1^1| + |q_1^2| > (\bar{\gamma} - 4\beta)|\partial\Pi| > (2\zeta + \gamma)|\partial\Pi| > |t_1|$ . Likewise,  $|q_1^1| + |q_1^2| > |t_2|$ . We then obtain the estimates  $|p't_1q''| < |\partial\Delta|$  and  $|p''q't_2| < |\partial\Delta|$ . Therefore, by the induction hypothesis, there is a chain of  $R$ -transformations  $Lab(p') \rightarrow Lab((q'')^{-1}t_1^{-1})$  in which no word has length greater than

$$\begin{aligned} D\max(|p'|, |q''| + |t_1|) &\leq D\max(|p'|, |q''|) + D(2\zeta + \gamma)nk \\ &< D\max(|p'|, |q''|) + 3\gamma nk. \end{aligned}$$

Additionally, since  $q_2^2$  is smooth in  $\partial\Gamma_q$ , by theorem II.26 we have

$$|q_1^2| \geq \bar{\beta}|q_2^2| - |p_1^2| - |p_2^2| > \bar{\beta}\beta nk - 2\zeta nk > \frac{9}{10}\beta nk.$$

So, we have a chain of  $R$ -transformations

$$\begin{aligned} Lab(p't_1(q_1^2)^{-1}t_2p'') &\rightarrow Lab((q'')^{-1}t_1^{-1}t_1(q_1^2)^{-1}t_2p'') \\ &\rightarrow Lab((q'')^{-1}(q_1^2)^{-1}t_2p'') \end{aligned} \quad (\text{X.5})$$

in which no word has length greater than

$$\begin{aligned} &D\max(|p'|, |q''|) + 3\gamma nk + |t_1(q_1^2)^{-1}t_2p''| \\ &< D\max(|p'|, |q''|) + |p''| + (4\gamma + 4\zeta)nk + |q_1^2| \\ &\leq D\max(|p'|, |q''|) + |p''| + D|q_1^2| - 2\beta|q_1^2| + 5\gamma nk \\ &< D\max(|p'|, |q''|) + D|q_1^2| + |p''| + 5\gamma nk - \frac{9}{5}\beta^2 nk \\ &< D[\max(|p'|, |q''|) + |q_1^2| + |p''|]. \end{aligned} \quad (\text{X.6})$$

Clearly, if  $|p'| \geq |q''|$ , this last quantity is

$$D(|p'| + |q_1^2| + |p''|) \leq D(|p'| + |q_1^1| + |p''|) = D|p|.$$

On the other hand, if  $|p'| < |q''|$ , the last line of (X.6) is

$$\begin{aligned} D[|p'| + (|q''| - |p'|) + |q_1^2| + |p''|] &\leq D[|p'| + (|q'| - |p''|) + |q_1^2| + |p''|] \\ &= D(|p'| + |q'| + |q_1^2|) < D(|q''| + |q'| + |q_1^2|) = D|q|. \end{aligned}$$

In either case, no word in our chain has length exceeding  $D \max(|p|, |q|)$ .

Now, applying the induction hypothesis to the diagram with contour  $p''q't_2$ , we have a chain of  $R$ -transformations  $Lab(p'') \rightarrow Lab(t_2^{-1}(q')^{-1})$  in which no word has length greater than  $D \max(|p''|, |t_2| + |q'|) < D \max(|p''|, |q'|) + 3\gamma nk$ . So, there is a chain of  $R$ -transformations

$$\begin{aligned} Lab((q'')^{-1}(q_1^2)^{-1}t_2p'') &\rightarrow Lab((q'')^{-1}(q_1^2)^{-1}t_2t_2^{-1}(q')^{-1}) \\ &\rightarrow Lab((q'q_1^2q'')^{-1}) \equiv Lab(q^{-1}) \end{aligned} \tag{X.7}$$

in which no word has length exceeding

$$\begin{aligned} &|(q'')^{-1}(q_1^2)^{-1}t_2| + D \max(|q'|, |p''|) + 3\gamma nk \\ &< |q''| + D|q_1^2| - 2\beta|q_1^2| + 3\gamma nk + D \max(|q'| + |p''|) + 3\gamma nk \\ &\leq |q''| + D|q_1^2| + 6\gamma nk - \frac{9}{5}\beta^2nk + D \max(|q'|, |p''|) \\ &< D[|q''| + |q_1^2| + \max(|q'|, |p''|)]. \end{aligned}$$

If  $|q'| \geq |p''|$ , then this quantity is  $D(|q''| + |q_1^2| + |q'|) = D|q|$ . On the other hand, if  $|q'| < |p''|$ , then  $|q''| < |p'|$  (since  $|q''| - |p'| \leq |q'| - |p''|$ ). In this case,

$$D[|q''| + |q_1^2| + \max(|q'|, |p''|)] < D(|p'| + |q_1^1| + |p''|) = D|p|.$$

In either case, no word in our chain has length greater than  $D \max(|p|, |q|)$ .

Combining chains (X.4), (X.5), and (X.7), we obtain a chain of  $R$ -transformations  $w \equiv Lab(p) \rightarrow Lab(q^{-1}) \equiv v$  in which no word has length exceeding  $D \max(|p|, |q|) = D \max(|w|, |v|)$ .

Since every  $S$ -letter labelling an edge of the diagram appears in  $w$  or  $v$ , every word in the chain of  $R$ -transformations that we constructed consists only of  $S$ -letters that appear in  $w$  or  $v$ .  $\square$

**Corollary X.5.** *The function  $f(s) = (1 + 2\beta)s$  is a filling length function for any partial-Burnside presentation of  $B_C(S, n)$ .*

*Proof.* Suppose  $w(S) = 1$  in  $B_C(S, n)$ . Then, by proposition X.4, there is a chain of  $R$ -transformations reducing  $w$  to 1 with no word in the chain having length greater than  $(1 + 2\beta)|w|$ .  $\square$

**Theorem X.6.** *Suppose that words  $w$  and  $v$  are conjugate in  $B_C(S, n)$ . Then, there is a chain of  $R$ -transformations and cyclic shifts*

$$w \equiv u_0 \rightarrow u_1 \rightarrow \cdots \rightarrow u_k \equiv v$$

*in which no word has length exceeding  $(1 + 3\beta) \max(|w|, |v|)$ . Furthermore, we may choose the chain of  $R$ -transformations in such a way that each word consists only of  $S$ -letters that appear in  $w$  or  $v$ .*

*Proof.* If  $w = 1$  in  $B_C(S, n)$ , so does  $v$ , and the previous result guarantees that there is a chain of  $R$ -transformations

$$w \rightarrow 1 \rightarrow v$$

in which no word has length exceeding  $(1 + 2\beta) \max(|w|, |v|)$ , and each word consists only of  $S$ -letters that appear in  $w$  or  $v$ .

Suppose now that  $w \neq 1$  in  $B_C(S, n)$  (then  $v \neq 1$ ) and consider a reduced annular diagram  $\Delta$  (over an MPBP of  $B_C(S, n)$ ) whose contours are labelled by  $w$  and  $v^{-1}$ . By theorem II.56,  $\Delta$  is an  $A$ -map. By lemma IV.5, every  $S$ -letter appearing in the contour label of an  $R$ -cell of  $\Delta$  appears in  $w$  or  $v$ . Furthermore, if an  $S$ -letter labels an edge in  $\Delta$  but does not appear in  $w$ ,  $v$ , or the label of an  $R$ -cell, this  $S$ -letter can be eliminated from the diagram. Therefore, we may assume that every  $S$ -letter of  $\Delta$  appears in  $w$  or  $v$ . By lemma II.28, there is a path  $t$  joining the contours of  $\Delta$  with  $|t| < \gamma(|w| + |v|)$ . Cutting along  $t$ , we obtain a circular diagram whose contour label yields the equality  $w^* = \text{Lab}(t) \cdot v^* \cdot \text{Lab}(t)^{-1}$  between cyclic shifts  $w^*$  and  $v^*$  of  $w$  and  $v$ , respectively. Furthermore, every  $S$ -letter labelling an edge of  $\Delta$  appears in  $w$  or  $v$ . Applying theorem X.4, there is a chain of  $R$ -transformations  $w^* \rightarrow \text{Lab}(t) \cdot v^* \cdot \text{Lab}(t)^{-1}$  in which every word consists only of  $S$ -letters from  $w$  or  $v$  and no word has length exceeding

$$\begin{aligned} (1 + 2\beta) \max(|w|, |v| + 2|t|) &< (1 + 2\beta) [\max(|w|, |v|) + \gamma(|w| + |v|)] \\ &\leq (1 + 2\beta)(1 + 2\gamma) \max(|w|, |v|) < (1 + 3\beta) \max(|w|, |v|) \quad (\text{LPP } \gamma < \beta). \end{aligned}$$

Of course we can pass from  $\text{Lab}(t)v^*\text{Lab}(t)^{-1}$  to  $v^*$  using cyclic shifts and deletions, and these transformations do not increase word length or introduce new  $S$ -letters. Likewise, we can pass from  $w$  to  $w^*$  using only cyclic shifts, which preserve length and the occurrence of  $S$ -letters. So, the result follows.  $\square$

**Proposition I.10.** *In a partial-Burnside group  $G = B_C(S, n)$  of large, odd partial-exponent, the following are equivalent:*

- (1) *The set of words representing finite order elements of  $G$  is recursive.*
- (2) *The word problem is solvable in  $G$ .*
- (3) *The conjugacy problem is solvable in  $G$ .*

*Proof.* (1)  $\Rightarrow$  (3): First, we can write  $G \cong B_{C'}(S, n)$  where  $C'$  is the set of words in the alphabet  $S$  representing finite order elements of  $G$ . Words  $w$  and  $u$  are conjugate in  $B_{C'}(S, n)$  if and only if there is a chain of  $R$ -transformations and cyclic shifts  $w \rightarrow w_1 \rightarrow \cdots \rightarrow w_k \equiv v$  with  $|w_i| < (1 + 3\beta)\max(|w|, |v|)$ , by theorem X.6. Moreover, there is such a chain whose words contain only letters that appear in  $w$  or  $v$ . Given a pair of words  $w$  and  $v$ , we can list all chains of  $R$ -transformations and cyclic shifts in which no word has length greater than  $(1 + 3\beta)\max(|w|, |v|)$ , no word appears more than once, and every letter appears in  $w$  or  $v$ . If the set words representing finite order elements of  $G$  is recursive, then we can check whether or not each step is an  $R$ -transformation or cyclic shift.

(3)  $\Rightarrow$  (2): Let  $u$  and  $w$  be words over  $S$ . If the conjugacy problem is solvable, we can determine whether or not  $wu^{-1}$  is conjugate to 1 in  $G$ . But this is equivalent to determining whether or not  $wu^{-1} = 1$ .

(2)  $\Rightarrow$  (1): Let  $u$  be a word over  $S$ . Then  $u$  represents finite order element of  $G$  if and only if  $u^n = 1$ . If the word problem is solvable, we can determine whether or not  $u^n = 1$ . Thus, if the word problem is solvable, we can determine which words represent finite order elements.  $\square$

**Corollary X.7.** *In a partial-Burnside group of large, odd partial-exponent, the word problem is solvable if and only if the conjugacy problem is solvable.*

The next result was proven in [NA], but we include it here as well since it follows so readily from proposition I.10.

**Corollary X.8.** *The word and conjugacy problem are solvable in every free Burnside group  $B(S, n)$  of large, odd exponent. In particular, there is an algorithm to determine whether or not a word  $w$  in  $F_\infty$  belongs to the verbal subgroup  $v(F_\infty)$ , where  $v = x^n$ .*

*Proof.* In a free Burnside group, every word represents a finite order element. It follows immediately from proposition I.10 that the word and conjugacy problems are solvable in  $B(S, n)$ . Of course, determining whether a word is in the verbal subgroup  $v(F_\infty)$  is equivalent to determining if the word is equal to 1 in  $B(\infty, n)$ , and the latter problem is solvable.  $\square$

**Proposition I.11.** *For free Burnside groups of large, odd exponent, the word problem has linear nondeterministic space complexity and polynomial deterministic space complexity.*

*Proof.* According to corollary X.8, the algorithm outlined in proposition I.10 solves the word problem in free Burnside groups of large, odd exponent. Moreover, since free Burnside groups have a linear filling length function by corollary X.5, the algorithm has linear nondeterministic space complexity. By Savitch's theorem (see corollary 1.31 of [DK]), there is a deterministic algorithm of polynomial space complexity that solves the same problem.  $\square$

**Lemma X.9.** *Let  $G = \langle S | R \rangle$  be a group given by a graded presentation such that every reduced diagram is an  $A$ -map. If  $w = 1$  in  $G$  with  $|w| > 0$ , then there is a diagram  $\Delta$  over  $G$  with contour label  $w$  such that, for any vertex  $o$  in  $\Delta$ , there is a path  $t$  from  $o$  to  $\partial\Delta$  with  $|t| < \gamma|w|$ .*



*Proof.* Let  $\Delta_0$  be a circular diagram over  $G$  with contour label  $w$  having minimal type. Suppose  $\Delta_0$  contains a circular subdiagram  $\Gamma$  with contour consisting only of 0-edges. If  $\Gamma$  is not a 0-cell, we can replace  $\Gamma$  by a single 0-cell with contour consisting only of 0-edges. This yields a new circular diagram  $\Delta_1$  in which there are fewer cells. Moreover, we see that  $\Delta_1$  also has contour label  $w$  and has the same type as  $\Delta_0$ .

Since this procedure can reduce the number of cells only finitely many times, we can repeat it and arrive at a diagram  $\Delta$  with contour label  $w$ , the same type as  $\Delta_0$ , and in which 0-cells are the only subdiagrams whose contours are composed entirely of 0-edges. Since  $\Delta$  has minimal type among all circular diagrams over  $G$  with contour label  $w$ , it follows by II.39 that  $\Delta$  is reduced. By hypothesis, the reduced map  $\Delta$  is an  $A$ -map. Therefore,  $\Delta$  satisfies the hypotheses of lemma IX.5. Hence, by lemma IX.5, given a vertex  $o$ , there is a path  $t$  connecting  $o$  and  $\partial\Delta$  with  $|t| < \gamma|\partial\Delta|$ .  $\square$

**Definition X.10.** Let  $G$  be a group with presentation  $\langle S|R \rangle$ . We say that  $f$  is an isodiametric function for this presentation of  $G$  if, for each  $w = 1$  in  $G$ , there is a circular diagram with contour label  $w$  and diameter  $\leq f(|w|)$ .

**Proposition X.11.** Let  $G = \langle S|R \rangle$  be a graded presentation of a group such that every reduced diagram is an  $A$ -map. If  $w = 1$  in  $G$  and  $|w| \leq s$ , and  $\Delta$  is as in lemma X.9, then  $\text{diam}(\Delta) < (\frac{1}{2} + 2\gamma)s$ .

*Proof.* Let  $x, y$  be vertices of  $\Delta$ . By lemma X.9, there are vertices  $x', y'$  on  $\partial\Delta$  with both the distance from  $x$  to  $x'$  and the distance from  $y$  to  $y'$  less than  $\gamma|\partial\Delta|$ . Clearly, there is a path from  $x'$  to  $y'$  on  $\partial\Delta$  whose length does not exceed  $\frac{1}{2}|\partial\Delta|$ . Therefore, there is a path from  $x$  to  $y$  with length less than  $(\frac{1}{2} + 2\gamma)|\partial\Delta|$ .  $\square$

By theorem II.56, we have:

**Corollary X.12.** For an MPBP of  $B_C(S, n)$ , the function  $f(s) = (\frac{1}{2} + 2\gamma)s$  is isodiametric.

**Definition X.13.** Let  $v(x_1, \dots, x_k)$  be a word in  $F_\infty$ . A verbal diagram for  $v$  ( $v$ -diagram) is a diagram over the presentation

$$\langle x_1, x_2, \dots \mid v(X_1, \dots, X_k), X_i \in F_\infty \rangle$$

of the group  $F_\infty/v(F_\infty)$ .

**Definition X.14.** Let  $v = v(x_1, \dots, x_k)$  be a word in  $F_\infty$ . We say that  $f_v$  is a verbal isodiametric function for  $v$  ( $v$ -isodiametric function) if, for each  $w \in v(F_\infty)$ , there is a  $v$ -diagram  $\Delta$  with contour label  $w$ , and  $\text{diam}(\Delta) \leq f_v(|w|)$ .

**Corollary X.15.** The function  $f(s) = (\frac{1}{2} + 2\gamma)s$  is  $v$ -isodiametric for the word  $v = x^n$ .

*Proof.* Suppose  $w = w(S)$  is in  $v(F_\infty)$ . Then  $w = 1$  in  $B(S, n)$ . By corollary X.12, there is a diagram  $\Delta$  over an MPBP of  $B(S, n)$  with contour label  $w$  and diameter  $\leq (\frac{1}{2} + 2\gamma)|w|$ . Since all cells of positive rank in this diagram are labelled with an  $n$ -th power, it follows that  $\Delta$  is also a verbal diagram, and the result follows.  $\square$

## CHAPTER XI

### INFINITELY BASED PRODUCTS OF VARIETIES

Using techniques related to partial-Burnside groups, we investigate infinitely based products of varieties. First, we examine the product of a Burnside variety and the variety of groups of nilpotency class at most  $c$ . Then, we present some work (joint with Olshanskii) about the product of a Burnside variety with itself.

Before continuing, we recall the definition of a product of varieties. For our purposes, it suffices to consider products whose factors are defined by a single law. If  $\mathfrak{U}, \mathfrak{V}$  are varieties of groups defined by the laws  $U(x_1, x_2, \dots, x_k) = 1$  and  $V(x_1, x_2, \dots, x_j) = 1$ , respectively, then the product variety  $\mathfrak{U}\mathfrak{V}$  is the class of all groups that satisfy  $U(V_1, V_2, \dots, V_k) = 1$  whenever each  $V_i$  is an element of the verbal subgroup corresponding to the word  $V$  (i.e. each  $V_i$  is a product of values of the word  $V$ ).

#### XI.1 $\mathfrak{B}_n$ times a nilpotent variety

For each positive integer  $c$ , the class of groups that are nilpotent of class at most  $c$  forms a variety  $\mathfrak{N}_c$ . This variety can be specified by a law:  $\mathfrak{N}_c$  is the variety of groups satisfying the law  $[[\dots[x_1, x_2], x_3], \dots, x_{c+1}] = 1$ .

Our objective is now to show that  $\mathfrak{B}_n \mathfrak{N}_c$  is infinitely based if  $n$  is large and odd. Our arguments are modified from the proof of theorem 31.6 in [O91], and our result extends this theorem, considering a broader class of varieties (theorem 31.6 in [O91] treats the abelian variety - i.e. nilpotent of class 1) and also eliminating the need for  $n$  to be prime. Our result also partially extends a main result of [K74]. While Kleiman's arguments deal with products  $\mathfrak{B}_n \mathfrak{V}$  for any finitely based  $\mathfrak{V}$  and without the restriction that  $n$  must be large and odd, his arguments do not allow  $n$  to be squarefree.

Before we begin the argument, we fix just a little more notation: given a commutative ring  $J$  with identity and left, unitary  $J$ -modules  $U_1, \dots, U_m$ , we denote by  $L^m(U_1, U_2, \dots, U_m; J)$  the set of  $m$ -multilinear maps from  $\prod_{i=1}^m U_i$  to  $J$ .

**Definition XI.1.** Let  $U_1, \dots, U_m$  be left, unitary  $J$ -modules. If  $f : \prod_{j=1}^m U_j \rightarrow J$  is  $m$ -multilinear, we define its right linear map  $f_{\tau} : U_m \rightarrow L^{m-1}(U_1, U_2, \dots, U_{m-1}; J)$  by  $f_{\tau}(u_m)(u_1, \dots, u_{m-1}) = f(u_1, \dots, u_{m-1}, u_m)$ . We define the right rank of  $f$ , denoted  $rank_{\tau}(f)$ , to be

$$\min \{ |T| : f_{\tau}(U_m) \subset \langle T \rangle, T \subset L^{m-1}(U_1, \dots, U_{m-1}; J) \}.$$

It is useful to point out a basic property of right rank. If  $f, g : \prod_{j=1}^m U_j \rightarrow J$  are  $m$ -multilinear, then  $rank_{\tau}(f + g) \leq rank_{\tau}(f) + rank_{\tau}(g)$ .

Adapting a construction from [O91], we form a multiplicative group  $G_{(m,k,n)}$  consisting of all  $(m+1)$ -by- $(m+1)$  upper triangular matrices  $||\psi_1^{(1)}, \dots, \psi_m^{(1)}, \psi_1^{(2)}, \dots, \psi_{m-1}^{(2)}, \dots, \psi_1^{(m)}||$  with 1's on the diagonal and

whose  $(i, j)$ -entry  $\psi_i^{(j-i)}(z_i, z_{i+1}, \dots, z_{j-1})$  is  $(j-i)$ -multilinear over the ring  $\mathbb{Z}_n$  on a product of copies of  $\mathbb{Z}_n^k$ , for  $j > i$ . In symbols,

$$\|\psi_i^{(j)}\| = \begin{pmatrix} 1 & \psi_1^{(1)}(z_1) & \psi_1^{(2)}(z_1, z_2) & \cdots & \psi_1^{(m)}(z_1, \dots, z_m) \\ 0 & 1 & \psi_2^{(1)}(z_2) & \cdots & \psi_2^{(m-1)}(z_2, \dots, z_m) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \psi_m^{(1)}(z_m) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

**Lemma XI.2.** *Let  $X = \|\xi_i^{(j)}\|$  and  $Y = \|\psi_i^{(j)}\|$  be elements of  $G_{(m,k,n)}$ . Suppose that in  $X - I$  (resp.  $Y - I$ ) the  $(i, j)$ -entry is zero unless  $j - i \geq x$  (resp.  $j - i \geq y$ ). Then in  $[X, Y] - I$ , the  $(i, j)$ -entry is zero unless  $j - i \geq x + y$ . Moreover, the  $(i, i + x + y)$ -entry of  $[X, Y]$  is  $\xi_i^{(x)} \psi_{i+x}^{(y)} - \psi_i^{(y)} \xi_{i+y}^{(x)}$ .*

*Proof.* We can write  $X = I + A$  and  $Y = I + B$  with  $A$  and  $B$  strictly upper triangular. Then  $A$  and  $B$  are nilpotent, and we can write  $X^{-1} = \sum_{u \geq 0} (-A)^u$  and  $Y^{-1} = \sum_{v \geq 0} (-B)^v$  (n.b. these are finite sums). Therefore,

$$[X, Y] = (I + A)(I + B) \sum_{u,v \geq 0} (-A)^u (-B)^v = I + (AB - BA) + C,$$

where  $C$  is a sum of products, each of which contains the factor  $A$  at least twice and the factor  $B$  at least once, or vice versa. Consequently, the  $(i, j)$ -entry of  $C$  is zero unless  $j - i \geq \min(2x + y, x + 2y)$ . Thus, the  $(i, j)$ -entry of  $[X, Y] - I$  is zero unless  $j - i \geq x + y$ , as claimed. Additionally, a simple computation finds that the  $(i, i + x + y)$ -entry is as claimed.  $\square$

For the remainder of this section let  $v(x_1, \dots, x_c) = [[[\dots[x_1, x_2], x_3], \dots, x_{c-1}], x_c]$ , the law defining the variety  $\mathfrak{N}_{c-1}$  of groups of nilpotency class at most  $c - 1$ .

**Lemma XI.3.** *The verbal subgroup  $v(G_{(c,k,n)})$  of  $G_{(c,k,n)}$  consists precisely of the matrices  $\|0, 0, \dots, 0, \psi_1^{(c)}\|$ . Moreover, if  $\psi_1^{(c)}$  has right rank exceeding  $2^c j$ , then  $\|0, 0, \dots, 0, \psi_1^{(c)}\|$  is not a product of  $j$  or fewer values of  $v$ .*

*Proof.* By a repeated application of lemma XI.2, every value of  $v$  in  $G_{(c,k,n)}$  has the form  $\|0, 0, \dots, 0, \xi_1^{(c)}\|$ . Of course, a product of such elements has the same form, and every element of  $v(G_{(c,k,n)})$  has the form claimed. Furthermore, by a repeated application of lemma XI.2, every value of  $v$  is a sum of  $2^c$  products of linear forms. Since each product of linear forms has right rank at most 1, every value of  $v$  has a rank at most  $2^c$ . Therefore, if  $\psi_1^{(c)}$  has rank greater than  $2^c j$ , then  $\|0, 0, \dots, 0, \psi_1^{(c)}\|$  is not a product of  $j$  or fewer values of  $v$ .

Now, we will show that given linear forms  $f_1, f_2, \dots, f_c : \mathbb{Z}_n^k \rightarrow \mathbb{Z}_n$ ,  $\|0, 0, \dots, 0, \prod_{i=1}^c f_i(z_i)\|$  is a value of  $v$ . Form an upper triangular matrix  $X_j$  with 1's on the diagonal and with  $(j, j + 1)$ -entry  $f_j$ . All other entries of  $X_j$  are zero. Now, we claim that  $v(X_1, \dots, X_c) = \|0, 0, \dots, 0, \prod_{i=1}^c f_i(z_i)\|$ . This is an easy induction.

Let  $\psi_1^{(c)}$  be  $c$ -multilinear. Then  $\psi_1^{(c)}(z_1, \dots, z_{c-1}, \sum_{i=1}^k a_i e_i) = \sum_{i=1}^k a_i \psi_1^{(c)}(z_1, \dots, z_{c-1}, e_i)$ . Moreover, each  $\psi_1^{(c)}(z_1, \dots, z_{c-1}, e_i)$  is  $(c-1)$ -multilinear in its first  $c-1$  coordinates. Therefore, by induction, it follows that  $\psi_1^{(c)}$  is a linear combination of products of linear forms. Consequently,  $\|0, 0, \dots, 0, \psi_1^{(c)}\|$  is in  $v(G_{(c,k,n)})$ .  $\square$

**Lemma XI.4.** *Given an integer  $j$ , the group  $G_{(c,2^c j,n)}$  contains elements  $g_1^{(i)}, g_2^{(i)}, \dots, g_c^{(i)}$  for  $1 \leq i \leq j$  such that for  $t = 1, 2, \dots, n-1$ , the element  $(\prod_{i=1}^j v(g_1^{(i)}, g_2^{(i)}, \dots, g_c^{(i)}))^t$  is not a value of  $(\prod_{i=1}^{j-1} v(x_{(i-1)c+1}, \dots, x_{ic}))^m$  for  $m \in \mathbb{Z}$ .*

*Proof.* Set  $k = 2^c j$ . For  $q = 1, \dots, k$ , let  $f_q(z_1, \dots, z_{c-1})$  be linearly independent  $(c-1)$ -multilinear forms in which  $f_q(e_{i_1}^{(1)}, \dots, e_{i_{c-1}}^{(c-1)})$  is either 0 or 1, where  $e_{i_a}^{(a)}$  is a member of the standard basis of the  $a$ -th copy of  $\mathbb{Z}_n^k$ . Let  $\delta_i : \mathbb{Z}_n^k \rightarrow \mathbb{Z}_n$  be the linear form taking  $e_i^{(c)} \mapsto 1$  and all other elements of the standard basis to 0. Then  $\sum_{i=1}^k f_i(z_1, \dots, z_{c-1}) \delta_i(z_c)$  is a  $c$ -multilinear form which takes only the values 0 and 1 when elements of the standard basis are substituted for each coordinate. Now, for  $t = 1, \dots, n-1$ , the multilinear form  $t \sum_{i=1}^k f_i \delta_i$  has right rank  $k = 2^c j$ . However, by lemma XI.3, every value of the  $(1, c+1)$ -entry of  $\prod_{i=1}^{j-1} v(x_{(i-1)c+1}, \dots, x_{ic})$  has rank at most  $2^c(j-1)$ . The same bound clearly holds for powers.  $\square$

Let  $S = \{s_1, s_2, \dots\}$ .

**Lemma XI.5.** *No subword of any cyclic shift of the word  $W \equiv \prod_{i=1}^k v(s_{(i-1)c+1}, \dots, s_{ic})$  is freely equal to a proper power.*

*Proof.* Suppose to the contrary that a subword  $U$  of a cyclic shift of  $W$  is freely equal to  $Z^k$  for some reduced word  $Z$  and some  $k > 1$ . Note that each  $S$ -letter appearing in  $Z$  must appear at least  $k$  times (with the same exponent) in  $U$ . Since each of the letters  $s_{ic}$  appears just once with exponent 1 and just once with exponent  $-1$ , it follows that  $U$  does not contain  $s_{ic}^{\pm 1}$ . Thus,  $U$  is a subword of  $[[\dots [x_{(i-1)c+1}, x_{(i-1)c+2}], x_{(i-1)c+3}], \dots, x_{ic-1}]$  or its inverse, for some  $i$ . Our result follows by repeatedly applying the argument just given.  $\square$

Next, before we can make use of the partial-Burnside group  $B_{\langle v(F(S)) \rangle}(S, n)$ , we need to verify that  $\langle v(F(S)) \rangle$  is a partial-Burnside subset of  $F(S)$ . Of course  $\langle F(S) \rangle$  is a normal subgroup of  $F(S)$ . Therefore, all that remains is to say that  $F(S)/v(F(S))$  is torsion-free. For this, we can refer to [N.H]:

**Lemma XI.6.** (31.62 of [N.H]) *Free nilpotent groups are torsion-free.*

**Lemma XI.7.** *The word  $W \equiv \prod_{i=1}^k v(s_{(i-1)c+1}, \dots, s_{ic})$  is simple rank  $|W|$  in  $B_{\langle v(F(S)) \rangle}(S, n)$ .*

*Proof.* Suppose to the contrary that  $W$  is not simple in rank  $|W|$ . Since  $W \in \langle v(F(S)) \rangle$  has finite order in  $B_{\langle v(F(S)) \rangle}(S, n)$ ,  $W$  is conjugate to a power  $U^j$  of a period  $U$  where  $|U| < |W|$ . Therefore, there is a reduced annular diagram  $\Delta$  with contours  $p$  and  $q$  labeled by  $W$  and  $U^{-j}$ . By lemma II.44 and theorems II.56 and II.57,  $\Delta$  is an  $A$ -map and we may assume that  $q$  is smooth.

Now, we will show that  $r(\Delta) = 0$ . Suppose to the contrary that  $\Delta$  has positive rank. Then  $\Delta$  contains a  $\gamma$ -cell  $\Pi$ , by theorem II.24. Let  $\Gamma_p$  and  $\Gamma_q$  denote contiguities of  $\Pi$  to  $p$  and  $q$ , respectively, with  $(\Pi, \Gamma_p, p) + (\Pi, \Gamma_q, q) > \bar{\gamma}$ . Since  $q$  is smooth, we have  $(\Pi, \Gamma_q, q) < \bar{\alpha}$  by lemma II.21. Therefore,  $(\Pi, \Gamma_p, p) > \bar{\gamma} - \bar{\alpha} > \varepsilon$ . It follows by theorem II.25 that  $\Delta$  contains an  $R$ -cell  $\pi$  and a contiguity  $\Gamma$  of  $\pi$  to  $p$  with  $(\pi, \Gamma, p) > \varepsilon$  and  $r(\Gamma) = 0$ . But this means that a cyclic shift of  $Lab(p) \equiv W$  must contain at least a proper power (since  $\varepsilon n > 2$ ), contrary to lemma XI.5. Therefore,  $r(\Delta) = 0$ , as claimed.

However, lemma XI.5 immediately implies that  $W$  cannot be freely conjugate to a proper power  $U^j$ .  $\square$

**Lemma XI.8.** *Let  $W_k \equiv \prod_{i=1}^k v(s_{(i-1)c+1}, \dots, s_{ic})$ . Given an integer  $j$ , for  $t = 1, \dots, n-1$ ,  $W_j^t$  is not a value in  $B_{\langle v(F(S)) \rangle}(S, n)$  of any word of the form  $(\prod_{i=1}^{j-1} v(x_{(i-1)c+1}, \dots, x_{ic}))^m$ .*

*Proof.* It is easy to see that every element of  $\langle v(G_{(c, 2^c j, n)}) \rangle$  has order dividing  $n$ . Consequently, there is a homomorphism  $B_{\langle v(F(S)) \rangle}(S, n) \rightarrow G_{(c, 2^c j, n)}$  taking  $s_{(i-1)c+k} \mapsto g_k^{(i-1)}$  for  $1 \leq i \leq j$ ,  $1 \leq k \leq c$  and taking the rest of  $S$  to the identity  $||0, 0, \dots, 0||$ . By lemma XI.4, our result is immediate.  $\square$

By lemma XI.7, we may choose a set  $\mathfrak{R}$  of periods of  $B_{\langle v(F(S)) \rangle}(S, n)$  which contains  $W_k = \prod_{i=1}^k v(s_{(i-1)c+1}, \dots, s_{ic})$ . Let  $N$  be the kernel of the projection  $F \rightarrow B_{\langle v(F(S)) \rangle}(S, n)$ .

By theorem II.64,  $N/[F, N]$  is free abelian with basis consisting of the elements  $\bar{R} = R^n[F, N]$  for  $R \in \mathfrak{R}$ . Let  $L_k/[F, N]$  be the subgroup generated by  $W_k^{n^2}[F, N]$  and  $\bar{R}$  for  $R \in \mathfrak{R} - \{W_k\}$ . Then  $N/L_k \equiv \mathbb{Z}_n$  and  $F/L_k$  is a central extension of  $B_{\langle v(F(S)) \rangle}(S, n)$ . Moreover, in  $F/L_k$ ,  $W_k^{n^2} = 1$ ,  $W_k^n \neq 1$ , and  $R^n = 1$  if  $W_k \neq R \in \mathfrak{R}$ .

**Proposition XI.9.** *Let  $j$  be an integer. Then  $F/L_j$  satisfies the law  $(\prod_{i=1}^{j-1} v(x_{(i-1)c+1}, \dots, x_{ic}))^n = 1$  but not the law  $(\prod_{i=1}^j v(x_{(i-1)c+1}, \dots, x_{ic}))^n = 1$ .*

*Proof.* Let  $X_1, \dots, X_{cj}$  be elements of  $F/L_j$ . Let  $U = \prod_{i=1}^{j-1} v(X_{(i-1)c+1}, \dots, X_{ic})$ . If  $U \in N/L_j$ , of course  $U^n = 1$  since  $N/L_j$  has order  $n$ . Otherwise, in  $B_{\langle v(F(S)) \rangle}$ ,  $U$  is conjugate to a power  $A^\ell$  of a period  $A$ , with  $1 \leq \ell < n$ . By lemma XI.8,  $A \neq W_j$ . Consequently,  $A^n = 1$  in  $F/L_j$ . Now, in  $F/L_j$ , we may write  $U = AZ$ , with  $Z \in N/L_j$ . Since  $Z$  is central and  $Z^n = 1$ , it follows that  $U^n = A^n Z^n = 1$  in  $F/L_j$ . Thus,  $F/L_j$  satisfies the law  $(\prod_{i=1}^j v(x_{(i-1)c+1}, \dots, x_{ic}))^n = 1$ . Of course we have already pointed out that  $W_j^n \neq 1$  in  $F/L_j$ .  $\square$

We have now proven the following theorem:

**Theorem I.12.** *Let  $\mathfrak{N}_c$  be the variety of groups of nilpotency class at most  $c$ . Then for  $n$  sufficiently large and odd, the product variety  $\mathfrak{B}_n \mathfrak{N}_c$  is not finitely based.*

## XI.2 $\mathfrak{B}_p \mathfrak{B}_p$

In this section, we consider a large prime  $p$  (which plays the role that  $n$  has played up to this point). The basic outline for our proof is the same as that given in the previous section. Let  $v(x) = x^p$ . For each integer  $m$ , we first construct a group in which there is an element that is a value of the word  $x_1^p x_2^p \cdots x_{m+1}^p$  but not a value of the word  $x_1^p x_2^p \cdots x_m^p$ . For this, we use the group  $G_{(p,m,p)}$  constructed in the previous section.

**Lemma XI.10.** *Every  $p$ -th power in  $G_{(p,m,p)}$  has the form  $||0, 0, \dots, 0, \psi||$ , where  $\psi$  is a product of linear  $p$  linear forms on  $\mathbb{Z}_p^m$ . The group  $G_{(p,m,p)}$  has exponent  $p^2$ , and the verbal subgroup  $\langle v(G_{(p,m,p)}) \rangle$  has exponent  $p$ . Additionally, the verbal subgroup corresponding to  $v$  is central.*

*Proof.* For an arbitrary element  $A$  of  $G_{(p,m,p)}$ , we have  $A = I + B$  where  $B$  is strictly upper triangular. The  $A^p = \sum_{i=0}^p \binom{p}{i} B^i = I + B^p$  since  $\binom{p}{i}$  is divisible by  $p$  for  $0 < i < p$ . Thus,  $A^p = ||0, 0, \dots, 0, \psi||$  where  $\psi$  is a product of the linear forms just above the diagonal, and  $A^{p^2} = ||0, 0, \dots, 0, p\psi|| = ||0, 0, \dots, 0||$ . Consequently  $G_{(p,m,p)}$  has exponent  $p^2$ . It is also clear now that the verbal subgroup corresponding to  $v$  has exponent  $p$ . Furthermore, from the form of  $p$ -th powers, it is clear that every  $p$ -th power is central; thus, the verbal subgroup corresponding to the word  $v$  is central.  $\square$

**Lemma XI.11.** *The verbal subgroup  $\langle v(G_{(p,m,p)}) \rangle$  of  $G_{(p,m,p)}$  contains a cyclic subgroup containing no nontrivial value of the word  $x_1^p x_2^p \cdots x_m^p$ .*

*Proof.* First, we count linear forms: there are  $p^m$  linear forms on  $\mathbb{Z}_p^m$ . Therefore, there are at most  $p^{mp}$  products of  $p$  linear forms. Making use of lemma XI.10, there are at most  $p^{m^2 p}$  elements of the form  $x_1^p x_2^p \cdots x_m^p$ .

On the other hand, the vector space of  $p$ -multilinear forms has dimension  $m^p$  over  $\mathbb{Z}_p$ . Therefore, the verbal subgroup has order  $p^{m^p}$  and consequently  $\frac{p^{m^p} - 1}{p - 1} > p^{m^p - 2}$  cyclic subgroups of order  $p$ . Since  $m^p - 2 > m^2 p$ , it follows that the verbal subgroup contains an element such that  $g^l$  is not a value of the word  $x_1^p x_2^p \cdots x_m^p$  for  $l = 1, 2, \dots, p - 1$ .  $\square$

Now, fix an integer  $m > 1$ . Let  $G = G_{(p,m,p)}$  and  $g = g_1^p g_2^p \cdots g_{m+1}^p$  an element such that  $g^l$  is not a value of the word  $x_1^p \cdots x_m^p$  for any  $l = 1, 2, \dots, p - 1$ . Consider an epimorphism  $f : F \rightarrow G$  of a free group onto  $G$ , and let  $H = F^p$ . By lemma XI.10,  $f(H)$  is central in  $G$ . Of course the subgroup  $H$  of the free group  $F$  is itself free; say that  $S$  is a basis of  $H$ . We define a graded presentation for a torsion group by specifying relators among the words over  $S$ . At each stage, we choose a collection of periods in the usual way; however, we alter the way we form relations from periods. When we choose a period  $A$ , we include a relation  $A^{p^2}$  if  $f(A)$  is a nontrivial element of  $\langle g \rangle$ . Otherwise, we include the relation  $A^p$ . Let  $N$  denote the normal closure of the set of relators. Next, we wish to show that  $N$  is normal in  $F$ .

**Lemma XI.12.** *An arbitrary element  $B$  of  $H$  represents an element of order  $p^2$  in  $H/N$  if  $f(B)$  is a nontrivial element of  $\langle g \rangle$ . Otherwise,  $B^p = 1$  in  $H/N$ .*

*Proof.* First,  $B$  is conjugate in  $H/N$  to a power of a period  $A$ ; say  $B = ZA^kZ^{-1}$ .

If  $f(B)$  is a nontrivial element of  $\langle g \rangle$ , then so is  $f(A^k)$ , since  $\langle g \rangle$  is central. Now, we can write  $g = f(A^k)^j = f(A^j)^k$  for some  $j$  relatively prime to  $p$ . Since  $g$  is not a  $p$ -th power in  $G$ , it follows that  $p$  does not divide  $k$ . Thus,  $f(A)$  is a nontrivial element of  $\langle g \rangle$ , and  $A$  has order  $p^2$ , by proposition III.8 (technically, by a slight modification in which not all defining relations have the same exponent - all the exponents must simply be sufficiently large of odd). Thus,  $B$  has order  $p^2$  as well.

If  $f(B)$  is not a nontrivial element of  $\langle g \rangle$ , there are two possibilities. First, if  $f(A)$  is a nontrivial element of  $\langle g \rangle$  and  $p|k$  then  $A$  has order  $p^2$  and so  $(A^k)^p = 1$ . Thus,  $B^p = 1$  as well. The other possibility is that  $f(A)$  is not a nontrivial element of  $\langle g \rangle$ , in which case  $A$  has order  $p$ . In this case,  $B^p = 1$ .  $\square$

This lemma implies that  $N$  is normal in  $F$ . Suppose  $x \in N$ . Then  $x$  is a product of some elements of the forms  $zy^{p^2}z^{-1}$  and  $zw^pz^{-1}$  where  $f(y)$  is a nontrivial element of  $\langle g \rangle$  and  $f(z)$  is not a nontrivial element of  $\langle g \rangle$ . Since  $H$  is normal in  $F$ , every  $F$ -conjugate of  $zy^{p^2}z^{-1}$  and  $zw^pz^{-1}$  are in  $H$ . Now, since  $f(H)$  is central, the preceeding lemma guarantees that  $F$ -conjugates of  $zy^{p^2}z^{-1}$  and  $zw^pz^{-1}$  remain in  $N$ .

Finally, we claim that  $F/N$  satisfies the law  $(x_1^p x_2^p \cdots x_m^p)^p = 1$  but not the law  $(x_1^p \cdots x_{m+1}^p)^p = 1$ .

It is immediate that  $F/N$  does not satisfy the law  $(x_1^p \cdots x_{m+1}^p)^p = 1$ . This is because  $g$  has a preimage  $g^*$  of order  $p^2$  in  $H/N$  by the previous lemma, and  $g^*$  has a preimage in  $F/N$ , which cannot be a product of  $m$  or fewer  $p$ -th powers (since  $g$  is not). On the other hand, a value of  $x_1^p \cdots x_m^p$  in  $F/N$  is necessarily in  $H/N$ ;  $f$  does not map this element to a nontrivial power of  $g$ , so this element has order dividing  $p$ .

We have proven:

**Theorem I.13.** *If  $p$  is a large prime, then  $\mathfrak{B}_p \mathfrak{B}_p$  does not have a finite basis of laws.*

## BIBLIOGRAPHY

- [A70] S. I. Adian, Infinite irreducible systems of group identities, Dokl. Akad. Nauk SSSR 190 (1970), 499-501; translation in Soviet Math. Dokl. 11 (1970), 113-115.
- [A75] S. I. Adian, The Burnside problem and identities in groups, Nauka, Moscow 1975.
- [A86] V. S. Atabekian, On simple infinite groups with identity #5381-B86, VINITI, Moscow, 1986 (this is kept in the Depot of VINITI, Moscow, and is available upon request).
- [A87] V. S. Atabekian, Simple and free periodic groups, Vestnik. Moskov. Univ. Ser. I Mat. Mekh. no. 6 (1987), 76-78.
- [Br] R. M. Bryant, Some infinitely based varieties of groups, J. Austral. Math. Soc. 16 (1973), 29-32.
- [Bu02] W. Burnside, On an unsettled question in the theory of discontinuous groups, Quart. J. Math. 33 (1902), 230-238.
- [Bu05] W. Burnside, On criteria for the finiteness of the order of a group of linear substitutions, Proc. London Math. Soc. (2) 3 (1905), 435-440.
- [DK] Ding-Zhu Du and Ker-I Ko, Theory of Computational Complexity, Wiley-Interscience Publ., New York, 2000.
- [Ge] S. M. Gersten, Isoperimetric and isodiametric functions, In G. Niblo and M. Roller (eds.): Geometric group theory, I, London Math. Soc. Lecture Note Series, vol. 181, Cambridge Univ. Press.
- [Ge91] S. M. Gersten, Isodiametric and isoperimetric inequalities in group extensions, preprint, Univ. of Utah 1991.
- [GeR] S. M. Gersten and T. R. Riley, Filling length in finitely presentable groups, Geometriae Dedicata, 92 (2002), 41-58.
- [Go] E. S. Golod, On nil-algebras and residually finite groups, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 273-276.
- [Gr] M. Gromov, Hyperbolic groups, in: Essays in Group Theory (S. M. Gersten, ed.), M.S.R.I. Pub. 8, Springer 1987, 75-263.
- [GK] C. K. Gupta and A. Krasilnikov, The Finite Basis Question for Varieties of Groups - Some Recent Results, Illinois J. of Math., 47 (2003), 1/2, 273-283.
- [H] M. Hall Jr., Solution of the Burnside Problem for Exponent Six, Illinois J. of Math. 2 (1958), 764-786.



- [I94] S.V. Ivanov, The free Burnside groups of sufficiently large exponents. *Internat. J. Algebra Comput.* 4 (1994), no. 1-2, ii+308 pp.
- [I03] S.V. Ivanov, On subgroups of free Burnside groups of large odd exponent, *Illinois J. Math.* 47, no. 1-2 (2003), 299-304.
- [K73] Ju. G. Kleiman, The basis of a product variety of groups I, *Izv. Akad. Nauk SSSR Ser. Mat.* 37 (1973), 95-97; translation in *Math. USSR Izv.* 37 (1973), 91-94.
- [K74] Ju. G. Kleiman, The basis of a product variety of groups II, *Izv. Akad. Nauk SSSR Ser. Mat.* 38 (1974), 475-483; translation in *Math. USSR Izv.* 38 (1974), 481-489.
- [L] I. G. Lysenok. Infinite Burnside groups of even period. (Russian) *Izvestial Rossiyskoi Akademii Nauk Seriya Matematicheskaya*, vol. 60 (1996), no. 3, pp. 3-224; translation in *Izvestiya. Mathematics* vol. 60 (1996), no. 3, pp. 453-654.
- [LW] F. Levi, B. L. van der Waerden, *Über eine besondere Klasse von Gruppen*, *Abh. Math. Sem. Hamburg. Univ.* 9 (1933), 154-158.
- [N.B] B. H. Neumann, Identical relations in groups I, *Math. Ann.* 114 (1937), 506-525.
- [N.H] H. Neumann, *Varieties of Groups*, Springer-Verlag, 1967.
- [NA] P.S. Novikov and S.I. Adian, *On infinite periodic groups I, II, III*, *Izv. Akad. Nauk SSSR. Ser. Mat.* **32** (1968), 212-244; 251-524; 709-731.
- [O70] A. Yu. Olshanskii, The finite basis problem for identities in groups, *Izv. Akad. Nauk SSSR Ser. Mat.* 34 (1970), 376-384; translation in *Math. USSR Izv.* 4 (1970), 381-389.
- [O91] A. Yu. Olshanskii, *The Geometry of Defining Relations in Groups*, Kluwer Academic Publishers, 1991.
- [O12] A. Yu. Olshanskii, Space functions of groups, *Trans. Amer. Math. Soc.*, 364 (2012), 4937-4985.
- [OOS] A. Yu. Olshanskii, D. V. Osin, M. V. Sapir, Lacunary Hyperbolic Groups, With an appendix by M. Kapovich and B. Kleiner, *Geom. and Topol.* 13 (2009), no. 4, 2051-2140.
- [OS] A. Yu. Olshanskii, M. V. Sapir, Embeddings of relatively free groups into finitely presented groups, *Contemporary Mathematics*, 264, 2000, 23-47.
- [P] P. Papasoglu, Strongly geodesically automatic groups are hyperbolic, *Inventiones Mathematicae*, 121 (1995), 323-334.
- [S] I. Schur, *Über Gruppen periodischer substitutionen*, *Sitzungsber. Preuss. Akad. Wiss.* (1911), 619-627.

- [V] M. R. Vaughan-Lee, Uncountably many varieties of groups, Bull. London Math. Soc. 2 (1970), 280-286.